Understanding Annuities.

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Some Algebraic Terminology.

- We recall some terms and calculations from elementary algebra.
- A finite sequence of numbers is a function of natural numbers $1, 2, \cdots, n$. Thus, the formula $a_k = 2k + 1$ for $k = 1, 2, \cdots, 10$ describes a sequence $3, 5, 7, 9, 11, 13, 15, 17, 19, 21$.
- We may also let a sequence run out to infinity as in $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$. Here the sequence can also be described as $\frac{1}{n}$ where $n = 1, 2, \cdots$.
- A sequence may also be called a progression. Two progressions are important, the Arithmetic Progression and the Geometric Progression.

A.P. and G.P.

- Arithmetic progression: This is a sequence which has a starting number a and successive numbers are obtained by adding a number d (called the common difference.)
  Thus, its n-th term is $a + (n-1)d$.
  Example: Take $a = 3, d = 4$. The sequence is

$$3, 7, 11, 15, 19, \cdots, 3 + 4(n-1), \cdots.$$

  The n-th term can be better written as $4n - 1$.
- Geometric progression: The geometric progression has a starting number a and successive terms are obtained by multiplying by a common ratio r.
  Thus, its n-th term is $ar^{(n-1)}$.
  Example: Take $a = 2$ and $r = \frac{1}{2}$. The sequence is:
  $2, 1, \frac{1}{2}, \frac{1}{4}, \cdots, \frac{2}{2^{(n-1)}}, \cdots$. Note that the n-th term is better written as $\frac{1}{2^{(n-2)}}$.

Arithmetic Series.

- We need the formula for the sum of terms in A.P.
- The sum of the A.P.

$$a, a + d, a + 2d, \cdots, a + (n-1)d$$

  is called an Arithmetic Series and is written as
  $\displaystyle\sum_{k=1}^{n}(a + (k-1)d)$.
- Its sum is given by the formula:

$$\sum_{k=1}^{n} a + (k-1)d = n\frac{a + a + (n-1)d}{2} = n\left(a + \frac{n-1}{2}d\right).$$

  An alternate way to remember it is (number of terms) $\cdot$ (average of the first and the last term).

Geometric Series.

- We need the formulas for the sum of terms in G.P.
- The sum of the G.P.

$$a, ar, ar^2, \cdots, ar^{(n-1)}$$

is called a Geometric Series and is written as $\sum_{k=1}^{n}(ar^{(k-1)})$.

- Its sum is given by the formula:

$$\sum_{k=1}^{n}(ar^{(k-1)}) = a\left(\frac{r^n - 1}{r - 1}\right) = a\left(\frac{1 - r^n}{1 - r}\right).$$

- If $|r| < 1$, then we can make sense of the formula even for an infinite G.P. and write; $\sum_{k=1}^{\infty}(ar^{(k-1)}) = a\left(\frac{1}{1-r}\right)$.

Basic Annuity.

- What is an annuity? An annuity is a combination of investments (or payments).
- For convenience,we assume the following conditions which are valid in most practical situations.
- A fixed amount R is invested exactly m times a year. This gives exactly m periods in a year and each is $\frac{1}{m}$-th part of the year.
- Each payment is made at the end of its period.
- The payments are made for a period of t years and thus the number of payments is exactly $mt = n$.
- For each period, the interest rate is the same $r\%$ annual and thus in each period, the interest earned by 1 dollar is exactly $\frac{r}{m} = i$. This is called the periodic rate.

Basic Annuity Formula.

- With the notation as explained above, how much money will be accumulated by making a periodic investment of R dollars at the end of each of the n periods when the periodic rate is i and the interest is compounded in each period?
- The answer comes out as a geometric series. Here is how we reason it out.
- The payment at the end of the first period is compounded for $(n-1)$ periods and hence becomes worth $R(1+i)^{(n-1)}$.
- The payment at the end of the second period is compounded only for $(n-2)$ periods and becomes worth $R(1+i)^{(n-2)}$.
- Continuing, the very last payment is worth $R(1+i)^{(n-n)} = R$. In other words, it acquires no interest!
- Adding up the terms in reverse, $S = R + R(1+i) + \cdots + R(1+i)^{(n-1)}$, or $S = R\frac{(1+i)^n-1}{(1+i)-1} = R\frac{(1+i)^n-1}{i}$.

A homework problem.

- If you invest \$300 per month at 5.9% compounded monthly, how much will your investment be worth in 25 years?
- You notice that you are given $R = 300$, $r = 5.9\%$ and $t = 25$.
- You deduce that $i = \frac{r}{m} = \frac{5.9}{1200}$ and $n = (25) \cdot (12) = 300$. You want to find S.
- We use the formula: $S = R\left(\frac{(1+i)^n - 1}{i}\right)$ to get \$204728.15.

Another homework problem.

- How much did you invest each month at 6.20% compounded monthly, if 25 years later the investment is worth $\$178,679.36$?
- You notice that you are not given R, but you know $r = 6.20\%$ and $t = 25.$ You also know $S = 178,679.36$.
- You deduce that $i = \frac{r}{m} = \frac{6.2}{1200}$ and $n = (25) \cdot (12) = 300$.
- We use the formula to write: $178679.36 = R\left(\frac{(1+i)^n - 1}{i}\right)$.
- The quantity $\frac{(1+i)^n - 1}{i}$ evaluates to 714.717547.
- Using this value, we get: $R = \frac{178679.36}{714.7175477} = 249.9999623$.
- So, $250 is a reasonably accurate answer.

Present Value of an Annuity.

- Often, the periodic investments are just payments - like mortgage - against borrowed funds. What is the relation between the periodic payment R and the borrowed amount P, when the interest rate is $r\%$ and the payment is m times a year?
- As usual, we let i be the periodic rate and n the number of periods or the total number of payments.
  Think like the lender and find out what single investment of P dollars would yield the same accumulation in same number of years and same rate.
- This gives us the equation: $P(1+i)^n = S = R\frac{(1+i)^n - 1}{i}$ and thus the formula: $P = R\frac{(1+i)^n - 1}{i(1+i)^n} = R\frac{1-(1+i)^{(-n)}}{i}$.
  This gives the needed formula $R = P\frac{i}{1-(1+i)^{(-n)}}$.

Using the Annuity Formulas.

- We now have the basic formulas needed to answer all questions about periodic investments or payments.
- Example of a Trust Fund If a trust is set up so that you take 6 years to travel and pursue other interests. Suppose that you will make bi-weekly withdrawals of $\$2,000$ from a money market account that pays 4.00% compounded bi-weekly. How much should the fund be?
- **Answer:** Imagine the trust fund to be a lender and your withdrawls as mortgage payments to you. Thus, we use the formula:
  $P = R\frac{1-(1+i)^{(-n)}}{i}$. Here $R = 2000$, $i = \frac{4}{2600}$ and $n = 26 \cdot 6 = 156$.
  The formula yields 277195.1659 or $\$277,195.17$.

More Examples of Annuities.

- Sinking Fund. This means a fund set up with periodic investments to be sunk or used up at the end of the n periods.
- Example. You plan on buying equipment worth $30,000$ dollars in 3 years. Since you firmly believe in not borrowing, you plan on making monthly payments into an account that pays 4.00% compounded monthly . How much must your payment be?
- You have to find out the value of R, but know that S, the expected accumulation is $30,000$ with $t = 3$ and $r = 0.04$.
- Moreover $m = 12$ (from the word monthly!!) and hence $i = \frac{0.04}{12} = .003333$ and $n = 12 \cdot 3 = 36$.
- Using $S = R\frac{(1+i)^n-1}{i}$ we get
  $R = 30000\left(\frac{i}{(1+i)^n-1}\right) = 785.7195502$.
- Thus, the reported answer is 785.72 which actually yields $\$30000.02$.

Continued Examples.

- About Accuracy. In the above calculation, the evaluation of

$$\frac{(1+i)^n - 1}{i} = \frac{(1+0.003333)^{36} - 1}{0.003333}$$

is involved. If you calculate this and divide into 30000, you need to keep many digits of accuracy. Try various approximations to see how to get the most accurate answer (to the penny).

- You will find that you need to keep at least four accurate decimal places the the first answer.
- Thus, as a general principle, in these problems, you should not copy down intermediate answers, but store and recall them, so that maximum accuracy is maintained.

Further Examples of Annuity.

- As another example, consider this problem.
- If you can afford a monthly payment of \$1010 for 33 years and if the available interest rate is 4.10%, what is the maximum amount that you can afford to borrow?
- You note that $R = 1010$, $i = \frac{r}{m} = \frac{0.041}{12}$ and $m = 12$ with $t = 33$, so that $n = 12 \cdot 33 = 396$.
- But you don't want S, the future accumulation! You want the money now, to be paid back over the years. So, you use the formula for P, the present value.
- Thus, you evaluate:
  $P = R\frac{1-(1+i)^{(-n)}}{i} = 1010 \cdot 216.8603683 = 219028.97.$
  Note that due to the large numbers involved, your fraction needs 10 digit accuracy!
- Thus, the hardest part is always to figure out which formula is appropriate!