More on Probability.

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Probability in Real Life.

- Experimental Probability:
  In real life situation, the sample spaces are huge and we don't even know them very well.
  So, it is important to be able to identify events properly.
  Moreover, usually we cannot calculate a probability, but can only estimate it.
- For instance, we consider the situation of persons in U.S. being for the Stimulus Package (F) or against it (A) or undecided (U).
- We can imagine a sample space S consisting of pairs (person, opinion) where the opinion is F or A or U. We will get a sample space S of size 3^n where n is the number of people in the united states, where, we only consider people who are relevant to the decision process! We define the event F to be the set of all pairs with opinion equal to F.

Example Continued.

- Then the probability of a person being for the stimulus package is $n(F)/(3^n)$. There is no way to actually carry out the necessary survey in a reasonable time or even in a meaningful way.
- So, statisticians resort to testing a few sample cases. This is why we have the word sample space! They select a few random persons, say d in number and ask how many of them are for the stimulus package. If they get a number f, then they will estimate the probability $P(F) = f/d$.
- Of course, this answer depends on the experiment (of surveying) and is only expected to be equal to - or close to the true probability $n(F)/(3^n)$. Such problems appear in section 7.2. Statisticians try to estimate the probability of the answer being correct by general theories and this is the basis of the estimated confidence levels or margin of error.

Probability Rules.

- If we know (or are confident about) the probabilities of certain events, it is possible to calculate the probabilities of related events.
  Here are some rules based on the set counting formulas and the conviction that the probability of any event E can be interpreted as $P(E) = n(E)/n(S)$.
- $P(E \bigcup F) = P(E) + P(F) - P(E \bigcap F)$.
- We say that two events E, F are mutually exclusive if they cannot happen together, or, in other words $P(E \bigcap F) = 0$. In this case, we get $P(E \bigcup F) = P(E) + P(F)$.
- $P(E^C) = 1 - P(E)$. This could be also explained that either E or E^C is always true, so $P(E \bigcup E^C) = 1$ and hence $1 = P(E) + P(E^C)$ since E and E^C are clearly mutually exclusive.

Probability Distribution.

- A simple event is an event with a single sample point.
- In Mathematical Probability we are acting as if all the sample points are equally likely. In real life, this is far from the case.
- A more realistic situation is that different sample points (or different simple events) might have different probabilities. Then the probability of an event is defined to be the sum of the probabilities of its constituent sample points.
- Consider an example of a class of 40 students with $4, 10, 18, 6, 2$ grades of A, B, C, D, E respectively.
  If we think of the five grades as our sample space, then the probability of getting A is not $\frac{1}{5} = 0.2$ but $\frac{4}{40} = 0.1$. Thus, the probability of getting a poor grade (D or E) is
  $\frac{6}{40} + \frac{2}{40} = 0.2$.
- The table or formula describing the probabilities of all simple events is called a probability distribution.

Probability Calculations.

- What is the probability of drawing a red card higher than 9 from a standard deck of 52 cards? (Don't count aces high.)
- **Answer:** The sample space clearly has 52 elements. Our event of a red card higher than 9 can be enumerated thus. In each suit there are 4 such cards and since we have two red suits, the total count by multiplication principle is
  $2 \cdot 4 = 8$.
  So, $P(E) = \frac{8}{52} = 0.1539$.
- What would be the probability of drawing a face card from a standard deck?
  Answer: If F is this event, then $n(F) = 3 \cdot 4$ so
  $P(F) = \frac{12}{52}$. (We are not counting aces as face cards.)
- What is the probability of not getting a face card?
  Answer: Easiest way to handle this is $1 - \frac{12}{52} = \frac{40}{52}$.

Continued Probability Calculations.

- Consider the situation where we draw three cards from a standard deck without replacement. Now the sample space has $C(52,3) = 22,100$ sample points.
- What would be the probability of getting at least one face card?
- **Answer:** If we call the event E and try to count $n(E)$ we soon realize that we have to make cases: getting one or two or three face cards.
  The calculation is clearly complicated.
- However, the count of choices without face cards is easy. It is just $C(40,3)$, the number of ways of choosing two cards from the forty non face cards.
- So,
  $$P(E^C) = \frac{C(40,3)}{C(52,3)} = \frac{40 \cdot 39 \cdot 38}{52 \cdot 51 \cdot 50} = \frac{38}{85} = 0.4471.$$
  So $P(E) = 1 - 0.4471 = 0.5529$.

Poker Hands.

- If you are playing a 5 card poker, what is the probability of getting a straight?
- **Answer:** Recall that a straight is defined as five cards in a sequence which are not from the same suit.
- The sample space has $n(S) = C(52,5) = 2,598,960$.
  We count the straight hands thus: We can start with a $1, 2, 3, \cdots, 10$ as the lowest card and fill up a straight. (Note that an ace is both a top and a bottom card!) Thus we have 10 cases of starting numbers to handle.
  For any choice of the starting number, there are $4^5 = 1024$ ways of filling up a sequence, when we don't worry about the "not same suit" condition. Now we remove the four sequences in the same suit to get 1020. Thus the total number of straights is $10 \cdot 1020 = 10200$.
- The desired probability is then $\frac{10200}{2598960} = 0.0039$.

Another Poker Hand.

- What is the probability of getting "three of a kind" which is defined as three cards with the same rank and the other two cards mismatched (of different ranks).
- We have thirteen different ranks and for each rank, there are four different three-of-a-kind triples of that rank. Thus a total of 52 triples are possible.
- To pick the remaining two cards, we choose two of the remaining ranks and these are $C(12, 2)$ choices. Moreover, we choose one of the four possible cards from these ranks. Thus we have a total of $52 \cdot C(12, 2) \cdot 4 \cdot 4 = 54, 912$.
- The desired probability is then: $\frac{54{,}912}{2{,}598{,}960} = 0.0211$.

Conditional Probability.

- Suppose there are events A, B under consideration. Suppose we know $P(A)$.
- If we later know that B has happened. This naturally affects the sample space, reducing it to those sample points belonging to the event B.
- What is the new probability of the event A?
- it is easy to see that in a mathematical set up, the answer is simply $n(A \bigcap B)/n(B)$.
- This is called the conditional probability of A given B and shall be denoted as $P(A|B)$.
- We note:

$$P(A|B) = \frac{n(A \bigcap B)}{n(B)} = \frac{P(A \bigcap B)}{P(B)} \text{ or } P(A \bigcap B) = P(A|B)P(B).$$

Example of conditional probability.

- Suppose we are playing a five card poker. Suppose that we have drawn an ace and a 9. What is the probability of getting a straight now?
- **Answer:** It is 0. We have to choose three more cards and no choice will fill up a straight joining these two cards.
- Now suppose we have drawn an ace and a 10 of different suits. What is the probability of a straight now?
- **Answer:** Now it is possible to fill in a Jack, queen and king to finish a straight.
  The sample space now consists of picking 3 cards from the remaining 50 cards, or $n(S) = C(50, 3)$.
  The three cards which will yield a straight can be made in $4 \cdot 4 \cdot 4 = 64$ ways by choosing one of the four jacks, one of the four queens and one of four kings.
  Thus the answer is $\frac{64}{C(50,3)} = \frac{4}{1225} = 0.0032653061$.

More types of Conditional Probability.

- Quite often, in order that a certain event happens, another event has to happen first. For instance, to get to class on time, you need to get up on time (with enough time to get ready) and then you need to get in a reasonable traffic.
- Given that the probability of getting up on time is 80% and then the probability of catching a good traffic pattern is 70%, then the probability of getting to class on time is calculated as the product $0.8 \cdot 0.7 = 0.56$ or 56%.
- What happens if you are a little late in getting up, say the probability of this is the remaining 20%.
  Perhaps, then the probability of getting a good traffic pattern is only 40%.
  Then the probability of getting to class on time is changed to: $0.2 \cdot 0.4 = 0.08$ or only 8%.

Reformulated discussion.

- Since, to get to class, you need to get up and fight with the traffic, you can now calculate the combined probability of getting to class on time as $0.56 + 0.08 = 0.64$ or 64%. Note that we simply added the two probabilities since the two events are mutually exclusive. If you get up in time, you have the first probability and if you get up late, you have the second.
- Let us formalize the above calculation as follows. Define events E and L as getting up early and getting up late. Define events G and B as getting a good traffic pattern and getting a bad traffic pattern.
- Thus, to get to class on time, you must have the event $E \bigcap G$ or $L \bigcap G$, so we want: $P((E \bigcap G) \bigcup (L \bigcap G))$. Note that the probability of getting a good traffic pattern $P(G)$ is not given to us at all, mainly because it changes based on the timing.

Continued Discussion.

- We do know that The probability of G given E is 70% while probability of G given L is only 40%.
- Our formulas say that then: The probability $P(E \bigcap G) = P(E) \cdot P(G|E)$ is then calculated as $0.8 \cdot 0.7 = 0.56$. Similarly $P(L \bigcap G) = 0.2 \cdot 0.4 = 0.08$.
- Finally, the two events $E \bigcap G$ and $L \bigcap G$ are mutually exclusive, so their probabilities are added!
- Food for thought: You can imagine a further possibility where you can get up so late that the traffic pattern does not matter! Think what could happen to the probability then. You should also calculate the probability of missing the class due to bad traffic!

Independent Events.

- If A, B are two events, we say that they are independent if the probability of A is the same whether B occurs or not. In other words, $P(A) = P(A|B)$.
- We recall that $P(A \bigcap B) = P(A|B) \cdot P(B)$ and thus the condition for independence comes out to be:

$$P(A \bigcap B) = P(A) \cdot P(B).$$

- Of course, in real life situations, the numbers are not likely to be exactly equal. Also, both sides being numbers less than 1, we need to have a stricter definition of considering them nearly equal.
- We accept the convention that the two events A, B are independent, if the corresponding percentage probabilities differ by less than 0.5.

Example of independence.

- Consider the following situation. In a factory 85 of the 260 managers and 185 of the 570 non managers got laid off.
- Are the events A : "being a manager" and B : "being laid off" independent?
- We estimate from the given data that $P(A) = \frac{260}{830}$ while $P(B) = \frac{270}{830}$. Also, $P(A \bigcap B) = \frac{85}{830}$.
- We check the condition if $\frac{85}{830} = \frac{260}{830} \cdot \frac{270}{830}$. The LHS evaluates to 0.1024 or 10.24%. The RHS evaluates to 0.1019 or 10.19%. The difference is 0.05%. This says that the events are independent by the agreed convention. However, if we change our convention, they may be declared dependent!
- This illustrates how a personal judgement and convention can enter the considerations!