Understanding monetary transactions.

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The Simple Interest.

- Interest is rent on borrowed money.
  We use the notation P for principal. This is the amount of money borrowed (or lent).
- We say r is the interest rate, if it is the agreed interest(rent) on one dollar per year. Of course, we change the name of the currency as appropriate!
  The rate is often quoted as something like 7% which means $r = \frac{7}{100} = 0.07$. Thus the words "per cent" mathematically mean the fraction $\frac{1}{100}$.
- We shall typically let t denote the period of lending in years and thus the interest I accumulated in t years on a principal of P dollars at a rate r is given by the formula:

$$I = Prt.$$

Simple Interest continued.

- Thus after the t years, the total amount owed is the original principal P plus the interest I and thus has the formula;

$$\text{Accumulation } A = P + Prt = P(1 + rt).$$

- If we know three of the four quantities A, P, r, t then we can find the fourth. We should learn to recognize what is given and what is unknown.
- **Example 1.** If you invest \$770.91 at 8% simple interest, how much will your investment be worth in 15 months?
- We note that $r = 8\%$ or $r = 0.08$. Also $P = 770.91$. We are given $t = 15$ months which must be converted to years and thus $t = \frac{15}{12} = 1.25$. We are looking for A, the net accumulated value of the investment. Hence

$$A = 770.91(1 + 0.08 \cdot 1.25) = 848.001. \text{ or } \$848.$$

More examples.

- **Example 2.** If you invest \$1127.52 and after 18 months it is worth \$1229.00, what simple interest rate, expressed as a percentage and rounded to .01, did you receive?
- We are given $P = 1127.52$, $t = \frac{18}{12} = 1.5$ and $A = 1229$. We want r. We recommend solving the formula $A = P(1 + rt)$ for r and then evaluating it. Thus:

$$r = \frac{\frac{A}{P} - 1}{t} = \frac{\frac{1229}{1127.52} - 1}{1.5}.$$

The answer comes out $r = 0.06000189206$. We multiply by 100 to make a per cent rate and report $r = 6\%$ after rounding.

- **Example 3.** If you invest \$1520.88 at 6% simple interest, after how many months, rounded to 0.01, will your investment be worth \$1657.00?
- We are given $P = 1520.88$, $r = 0.06$, $A = 1657$ and asked to find t. As before, we solve our formula for t and evaluate:

$$t = \frac{\frac{A}{P} - 1}{r} = \frac{\frac{1657}{1520.88} - 1}{0.06}.$$

This gives $t = 1.491680255$. Be sure to multiply by 12 to make months. So the answer is 17.90016306 or 17.9 months after rounding.

A More Complicated Example.

- **Example 4.** Homer won a prize in the lottery of \$3000, payable \$1500 immediately and \$1500 plus 4% simple interest payable in 260 days. Getting impatient, Homer sells the promissory note to Moe for \$1440 cash after 170 days. Using a nominal 360 day year, find the simple interest rate, rounded to 0.01, earned by Moe.
- This is a simple interest problem of finding r, but needs careful set up. If Homer were to patiently wait the 260 days, he would earn

$$A = P(1 + rt) = 1500(1 + 0.04 \cdot \frac{260}{360}) = 1543.333333 \text{ dollars} .$$

- From Moe's perspective, this is his A after a lending of \$1440 for a period of $260 - 170 = 90$ days. Thus for Moe, the calculated interest rate as in Example 2 is

$$r = \frac{\frac{A}{P} - 1}{t} = \frac{\frac{1543.33}{1440} - 1}{\frac{90}{360}} = 0.2870370361 = 28.7\%.$$

Such high rates are not uncommon for short term lenders!

The Greedy Lender.

- Suppose you lend somebody \$100 for a period of one year at 10% interest rate. You will receive an accumulated payback of \$110 at the end of the year.
  But if you demand a repayment in six months, you will be entitled to receive \$105. Now, suppose you lend this total amount back to the borrower, then using the usual formula with
  $P = 105, r = 0.10, t = 0.5$ we get $105(1 + 0.10 \cdot 0.5) = 110.25$ dollars!
  It is easy to see that the net formula is $100(1 + \frac{0.10}{2})^2$.
- Of course, you don't really want to carry out the transaction, just demand the money. This is called the accumulation by compounding every six months or twice a year!
  Thus a greedy lender can claim more money by simply "imagining" a transaction!

The Compound Interest.

- If one gets greedier and imagines compounding the interest m times a year then it is easy to see that in each of the m periods, we get the accumulation by multiplying the starting principal for that period by $(1 + \frac{r}{m})$ and thus the full formula for the interest after one year is:
$$A = P\left(1 + \frac{r}{m}\right)^m.$$
- It is useful to develop some **new notation.**
Assume that we are compounding m times each year. Thus in t years, we shall have mt periods of compounding and we define: Periodic interest rate $i = \frac{r}{m}$ and Total term of loan in periods $n = tm$.
- This gives us the formula for accumulated amount when we compound m times a year as:
$$A = P\left(1 + \frac{r}{m}\right)^{mt} = P(1 + i)^n.$$

This Greed has a Limit!

- Continuing our example of lending \$100 for one year at a rate of 10%. If we compound it m times a year, then we have the formula $A_m = 100\left(1 + \frac{0.10}{m}\right)^m$.
- We can calculate the accumulation for different values of m:

m	1	11	51	101
A_m	110.00	110.46717	110.50627	110.51162

- Thus, though increasing, it is not growing very fast. Indeed, using techniques of algebra it is possible to show that the limit of the quantity A_m as m goes to infinity is a famous function of mathematics, namely

$$\lim_{m\to\infty} P\left(1 + \frac{r}{m}\right)^m = P\exp(r).$$

Thus, even if we imagine infinite compounding, our accumulation for the above $P = 100, r = 10\%$ is only $100\exp(0.10) = 110.5170918$.

The Compound Interest Formulas.

- To summarize, we have the formula that for principal P, annual rate r, period t years and compounded m times a year, we have

$$A_m = P(1+i)^n \text{ where } i = \frac{r}{m},\ n = tm.$$

- We describe the idea of infinite compounding as continuous compounding. The accumulation if we compound continuously is given by the formula:

$$A_C = P\exp(rt).$$

- Typically, we just write A for accumulation, but mention the value of m in words.

Examples of Compound Interest.

- **Example 5.** If you invest \$5000.00 at 9% compounded bi-weekly, how much will your investment be worth in 8 years?
- We have $P = 5000, r = 0.09, t = 8$. The meaning of the phrase bi-weekly is that it is compounded once every two weeks or $m = 26$ using a nominal year of 52 weeks.
  We have $i = \frac{0.09}{26} = 0.003461538462$ and $n = 8 \cdot 26 = 208$.
  Thus

$$A = 5000(1 + 0.003461538462)^{208} = 10259.40275.$$

- Warning: It is crucial to learn good calculator techniques here, since if you don't keep enough accuracy for i, then the power calculation introduces a lot of error and multiplication by a large P makes a very inaccurate amount. One should try not to copy down intermediate results, but store and reuse them for better accuracy!

More Examples.

- **Example 6.** How much did you invest at 8% compounded bi-weekly if 15 years later the investment is worth $97000.00?
- If the investment is P then our formula gives:

$$97000 = P\left(1+\frac{0.08}{26}\right)^{(15\cdot 26)}$$

which can be solved for P as:

$$P = 97000\cdot\left(\left(1+\frac{0.08}{26}\right)^{(-15\cdot 26)}\right).$$

- This evaluates to 29269.71472. It is an excellent idea to double check that this value of P does generate the 97000 i.e.

$$29269.71472\left(1+\frac{0.08}{26}\right)^{15\cdot 26} = 97000$$

within reasonable accuracy! The computer answer is 96999.99712.

Effective Rate.

- Often, lending terms are described by different rates and different number of compoundings per year. It is necessary to be able to compare them to decide which is a better rate.
- One way to do this is to find an effective rate r_{eff}, which is defined as a simple interest rate which will give the same yield as the given scheme.
- Thus, if we invest one dollar at $r\%$ annual rate compounded m times a year, then our net yield is $(1 + \frac{r}{m})^m$ and if r_{eff} is to be the effective rate, then we have:

$$\left(1 + \frac{r}{m}\right)^m = 1 + r_{\mathit{eff}}$$

so we have the formula;

$$r_{\mathit{eff}} = \left(1 + \frac{r}{m}\right)^m - 1.$$

Example of Effective Rate.

- **Example 7.** Bank A is offering an interest rate of 6.60% compounded monthly, while bank B is offering an interest rate of 6.69% compounded quarterly.
  What are the effective rates of the two banks expressed as percents and for the investor, which bank offers the bettter rate?
- We apply the formula for the effective rate to get:
  The r_{eff} for bank A is: $\left(1 + \frac{0.066}{12}\right)^{12} - 1 = 0.06803356$
  and the r_{eff} for bank B is: $\left(1 + \frac{0.0669}{4}\right)^{4} - 1 = 0.06859714600.$
- The reported answers should be 6.80% and 6.86% respectvely, with bank B declared as having a better rate.
- Note that if the problem was about borrowing from the bank instead of investing, then bank A would be a better choice!!

Preview.

- Next, we study the concepts of progression or a sequence and a series (or their sum).
  Of special interest are the Arithmetic series and the Geometric series; a must study for all students of mathematics!
- Afterwards, we tackle a problem of annuity. Such problems have three types.
- We discuss how to borrow a large sum and pay back with periodic payments (mortgage).
- We discuss how much money can be withdrawn on a periodic basis from set up funds which are earning interest until drawn (sinking funds).
- We also dicuss how to build up future reserves by periodic saving.