

Understanding Annuities.

Ma 162 Spring 2009

Ma 162 Spring 2009

March 2009

Some Algebraic Terminology.

- We recall some terms and calculations from elementary algebra.
- A finite sequence of numbers is a function of natural numbers $1, 2, \dots, n$. Thus, the formula $a_k = 2k + 1$ for $k = 1, 2, \dots, 10$ describes a sequence 3, 5, 7, 9, 11, 13, 15, 17, 19, 21.
- We may also let a sequence run out to infinity as in $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$. Here the sequence can also be described as $\frac{1}{n}$ where $n = 1, 2, \dots$.
- A sequence may also be called a **progression**. Two progressions are important, the Arithmetic Progression and the Geometric Progression.

- Arithmetic progression: This is a sequence which has a **starting number** a and successive numbers are obtained by adding a number d (called the **common difference**.)

Thus, its n -th term is $a + (n - 1)d$.

Example: Take $a = 3, d = 4$. The sequence is

$$3, 7, 11, 15, 19, \dots, 3 + 4(n - 1), \dots$$

The n -th term can be better written as $4n - 1$.

- Geometric progression: The geometric progression has a **starting number** a and successive terms are obtained by multiplying by a **common ratio** r .

Thus, its n -th term is $ar^{(n-1)}$.

Example: Take $a = 2$ and $r = \frac{1}{2}$. The sequence is:

$$2, 1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{2}{2^{(n-1)}}, \dots$$

Note that the n -th term is better written as $\frac{1}{2^{(n-2)}}$.

Arithmetic Series.

- We need the formula for the sum of terms in A.P.
- The sum of the A.P.

$$a, a + d, a + 2d, \dots, a + (n - 1)d$$

is called an Arithmetic Series and is written as $\sum_{k=1}^n (a + (k - 1)d)$.

- Its sum is given by the formula:

$$\sum_{k=1}^n a + (k - 1)d = n \frac{a + a + (n - 1)d}{2} = n \left(a + \frac{n - 1}{2} d \right).$$

An alternate way to remember it is
(number of terms) \cdot (average of the first and the last term).

Geometric Series.

- We need the formulas for the sum of terms in G.P.
- The sum of the G.P.

$$a, ar, ar^2, \dots, ar^{(n-1)}$$

is called a Geometric Series and is written as $\sum_{k=1}^n (ar^{(k-1)})$.

- Its sum is given by the formula:

$$\sum_{k=1}^n (ar^{(k-1)}) = a \left(\frac{r^n - 1}{r - 1} \right) = a \left(\frac{1 - r^n}{1 - r} \right).$$

- If $|r| < 1$, then we can make sense of the formula even for an infinite G.P. and write;

$$\sum_{k=1}^{\infty} (ar^{(k-1)}) = a \left(\frac{1}{1 - r} \right).$$

Basic Annuity.

- What is an annuity? An annuity is a combination of investments (or payments).
- For convenience, we assume the following conditions which are valid in most practical situations.
- A fixed amount R is invested exactly m times a year. This gives exactly m periods in a year and each is $\frac{1}{m}$ -th part of the year.
- Each payment is made at the end of its period.
- The payments are made for a period of t years and thus the number of payments is exactly $mt = n$.
- For each period, the interest rate is the same $r\%$ annual and thus in each period, the interest earned by 1 dollar is exactly $\frac{r}{m} = i$. This is called the periodic rate.

Basic Annuity Formula.

- With the notation as explained above, how much money will be accumulated by making a periodic investment of R dollars at the end of each of the n periods when the periodic rate is i and the interest is compounded in each period?
- The answer comes out as a geometric series. Here is how we reason it out.
- The payment at the end of the first period is compounded for $(n - 1)$ periods and hence becomes worth $R(1 + i)^{(n-1)}$.
- The payment at the end of the second period is compounded only for $(n - 2)$ periods and becomes worth $R(1 + i)^{(n-2)}$.
- Continuing, the very last payment is worth $R(1 + i)^{(n-n)} = R$. In other words, it acquires no interest!
- Adding up in reverse, we have

$$S = R + R(1+i) + \cdots + R(1+i)^{(n-1)} = R \frac{(1+i)^n - 1}{(1+i) - 1} = R \frac{(1+i)^n - 1}{i}.$$

Present Value of an Annuity.

- Often, the periodic investments are just payments - like mortgage - against borrowed funds. What is the relation between the periodic payment R and the borrowed amount P , when the interest rate is $r\%$ and the payment is m times a year?
- As usual, we let i be the periodic rate and n the number of periods or the total number of payments.

We think like the lender and argue that he makes a single payment of P dollars and expects it to grow to the same sum as the periodic payments by the borrower over the same number of periods.

- This gives us the equation:

$$P(1+i)^n = S = R \frac{(1+i)^n - 1}{i}$$

and thus the formula:

$$P = R \frac{(1+i)^n - 1}{i(1+i)^n} = R \frac{1 - (1+i)^{-n}}{i}$$

This gives the needed formula

Using the Annuity Formulas.

- We now have the basic formulas needed to answer all questions about periodic investments or payments.
- **Example of a Trust Fund** If a trust is set up so that you take 6 years to travel and pursue other interests. Suppose that you will make bi-weekly withdrawals of \$2,000 from a money market account that pays 4.00% compounded bi-weekly. How much should the fund be?
- **Answer:** Imagine the trust fund to be a lender and your withdrawals as mortgage payments to you. Thus, we use the formula:

$$P = R \frac{1 - (1 + i)^{(-n)}}{i}.$$

Here $R = 2000$, $i = \frac{4}{2600}$ and $n = 26 \cdot 6 = 156$.

The formula yields 277195.1659 or \$277,195.17.

More Examples of Annuities.

- **Sinking Fund.** This means a fund set up with periodic investments to be sunk or used up at the end of the n periods.
- **Example.** You plan on buying equipment worth 30,000 dollars in 3 years. Since you firmly believe in not borrowing, you plan on making monthly payments into an account that pays 4.00% compounded monthly. How much must your payment be?
- You have to find out the value of R , but know that S , the expected accumulation is 30,000 with $t = 3$ and $r = 0.04$.
- Moreover $m = 12$ (from the word monthly!!) and hence $i = \frac{0.04}{12} = .003333$ and $n = 12 \cdot 3 = 36$.
- Using $S = R \frac{(1+i)^n - 1}{i}$ we get

$$R = 30000 \left(\frac{i}{(1+i)^n - 1} \right) = 785.7195502.$$

- Thus, the reported answer is 785.72 which actually yields \$30000.02.

Continued Examples.

- **About Accuracy.** In the above calculation, the evaluation of

$$\frac{(1 + i)^n - 1}{i} = \frac{(1 + 0.003333)^{36} - 1}{0.003333}$$

is involved. If you calculate this and divide into 30000, you need to keep many digits of accuracy. Try various approximations to see how to get the most accurate answer (to the penny).

- You will find that you need to keep at least four accurate decimal places the the first answer.
- Thus, as a general principle, in these problems, you should not copy down intermediate answers, but store and recall them, so that maximum accuracy is maintained.

Further Examples of Annuity.

- As another example, consider this problem.
- If you can afford a monthly payment of \$1010 for 33 years and if the available interest rate is 4.10%, what is the maximum amount that you can afford to borrow?
- You note that $R = 1010$, $i = \frac{r}{m} = \frac{0.041}{12}$ and $m = 12$ with $t = 33$, so that $n = 12 \cdot 33 = 396$.
- But you don't want S , the future accumulation! You want the money now, to be paid back over the years. So, you use the formula for P , the present value.
- Thus, you evaluate:

$$P = R \frac{1 - (1 + i)^{(-n)}}{i} = 1010 \cdot 216.8603683 = 219028.97.$$

Note that due to the large numbers involved, your fraction needs 10 digit accuracy!

- Thus, the hardest part is always to figure out which formula is appropriate!