Lecture on sections 1.3,1.4

Ma 162 Spring 2009

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Review: Basic Definitions.

- A linear function of one variable x is a function $f(x) = mx + c$ where m, c are constants. Its graph is a straight line, hence it is called linear.
  Example: $f(x) = 3x + 4$.
- A linear function of two variables x, y is of the form $f(x, y) = ax + by + c$. Its graph is a plane in three space.
  Example: $f(x, y) = 3x + 4y + 5$.
- A natural generalization is a linear function of n variables $f(x_1, x_2, \cdots, x_n) = a_1x_1 + a_2x_2 + \cdots a_nx_n + b$ where $a_1, a_2, \cdots, a_n, b$ are constants.
  Example: $f(x, y, z) = 3x + 4y - 5z + 7$.
- These functions are useful in many applications.
- What are examples of functions which are **not** linear?
  Example:
  $f(x) = x^2 + 3x, g(x, y) = x^3 - y^3, h(x, y, z) = xy + yz + zx.$

Some interesting Linear Functions

Here are some examples of real life functions which behave like linear functions.

- **Tax Calculations.** Typically, tax calculation on an income of x dollars is a linear function when x lies in a specific tax bracket. The function formula, however, changes with tax brackets. This is a good example of a step function which is defined by different formulas in different ranges of x values. A typical formula looks like

  $$t(x) = f + r(x - b)$$

  where x is assumed to be at least b, f is the fixed tax for income b and r is the tax on each dollar earned above b.

Tax Calculations continued.

Example: What is the tax on $\$49,000$, if tax is charged at 6% on all income above $\$15,000$?
Answer: Tax is $0.06(49000 - 15000) = 2040$ dollars.
Example continued: What is the tax if amounts above $\$50,000$ are charged at the rate of 7% and the income is $\$60,000$?
Answer:
Tax on $\$50,000$ by the the first formula is
$f = 0.06(50000 - 15000) = 2100$. Tax for the excess of
$\$(60000 - 50000)$ or $\$10,000$ is $0.07(10000) = 700$.
Thus, the net tax is $2100 + 700 = 2800$ dollars.

Depreciation.

- **Depreciation.** If an initial value b is to be depreciated to value zero in d years, then the depreciated value for any t between 0 and d is given by the formula:

$$v(t) = b - \frac{b}{d}t.$$

Example: If a car worth $\$45,000$ is to be depreciated to zero in 6 years, what is its value after 4 years.
Answer: Here $b = 45000$, $d = 6$ and $t = 4$.
So the formula gives $45000 - \frac{45000}{6}4 = 15000$.

Financial Functions

- **Review of Business Functions.** If x is the number of units sold or manufactured, then we have three natural functions associated with it.
- The cost function is $C(x) = cx + f$ where c is the production cost per unit and f is the fixed cost.
- The revenue function is $R(x) = px$ where p is the selling price of a unit.
- The profit function is then given by $P(x) = R(x) - C(x) = (p - c)x - f$.

Breakeven Analysis

We now describe how to determine the minimum production level which guarantees no net loss.

- **Breakeven Production.** A production level x is said to be break even if the net profit is zero, i.e.

$$P(x) = R(x) - C(x) = (p - c)x - f = 0.$$

- **Another viewpoint.** The breakeven production can be interpreted alternatively as follows.
  Consider the graphs of the revenue function $R(x)$ and the cost function $C(x)$ plotted on the same axes. Both graphs are lines.
  Then the break even production is simply the x-coordinate of the common point.

Example of Breakeven Analysis.

Example. Suppose that a certain toy can be manufactured for \$3.5 each and the cost of maintaining the workshop is \$21,000 per month.

If the toy is sold for \$5, determine the monthly breakeven production.

Answer: Let x denote the monthly production. Then the revenue function is $R(x) = 5x$. The cost function is $C(x) = 3.5x + 21000$. We find the common point of the graphs of $y = R(x)$ and $y = C(x)$, thus we solve:

$$5x = 3.5x + 20000 \text{ or } 1.5x = 21000.$$

The breakeven production is $x = \dfrac{21000}{1.5} = 14000$.

Intersecting lines

The above example motivates a review of techniques to intersect two lines.

We consider two linear equations $ax + by = c$ and $px + qy = r$ and discuss their common solutions.

- **Substitution method.** The most familiar method is to solve one of the equations for y and substitute the solution in the other to find the x-coordinate of the common point. Then use it and one of the equations to find y. **Example:** Solve

  $$E1: \quad 3x - y = 5 \text{ and } E2: \quad 2x + 3y = 7.$$

  Answer: Solving E1 for y, we get $y = 3x - 5$. Substitution in E2 gives: $2x + 3(3x - 5) = 7$ or $11x = 22$. This gives $x = 2$ and finally $y = 3x - 5 = 3(2) - 5 = 1$.

Continued Intersections of Lines.

- Thus, the intersection of the lines

$$E1: \quad 3x - y = 5 \text{ and } E2: \quad 2x + 3y = 7$$

  is given by $x = 2, y = 1$.
- **Comment.** Though easy to understand, the above method is not always the most efficient and it is helpful to learn other techniques as well.
- **Cramer's Rule.** This technique lets us write down the answer to a system of two linear equations in two variables by a formula.
  It can also generalize to several equations in several variables.

Determinants.

- **Determinant.** Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define its determinant:

$$\det(A) = ad - bc.$$

Sometimes this is also written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Example:

$$\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix} = (5)(4) - (-3)(2) = 26.$$

Cramer's Rule.

- Now we describe the Cramer's rule for solving two equations in two variables:

$$E1 : ax + by = c, \quad E2 : px + qy = r.$$

- Calculate the determinants:

$$\Delta = \begin{vmatrix} a & b \\ p & q \end{vmatrix}, \ \Delta_x = \begin{vmatrix} c & b \\ r & q \end{vmatrix}, \ \Delta_y = \begin{vmatrix} a & c \\ p & r \end{vmatrix}.$$

- Then the answers are:

$$x = \frac{\Delta_x}{\Delta} \text{ and } y = \frac{\Delta_y}{\Delta}.$$

Cramer's Rule: continued.

- Recall the answers:

$$x = \frac{\Delta_x}{\Delta} \text{ and } y = \frac{\Delta_y}{\Delta}.$$

- These are undefined when $\Delta = 0$ and in that case we have:
  - If one of Δ_x, Δ_y is non zero, then the system has no solution.
  - If all three determinants are zero then one equation is a multiple of another and we could have infinitely many solutions **unless** we happen to have an equation where all coefficients of variables are zero and some right hand side is non zero!

Example of Cramer's Rule.

- We redo the old example.
  Example: Solve
  $$E1: \quad 3x - y = 5 \text{ and } E2: \quad 2x + 3y = 7.$$
  Answer: We see that
  $$\Delta = \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} 5 & -1 \\ 7 & 3 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix}.$$
- Evaluation gives:
  $$\begin{aligned} \Delta &= (3)(3) - (-1)(2) &= 11 \\ \Delta_x &= (5)(3) - (-1)(7) &= 22 \\ \Delta_y &= (3)(7) - (5)(2) &= 11 \end{aligned}$$
- So the answers are:
  $$x = \frac{11}{11} = 1, \quad y = \frac{22}{11} = 2$$
  as alreday known!