

Lecture Simplex algorithm I

Ma 162 Spring 2009

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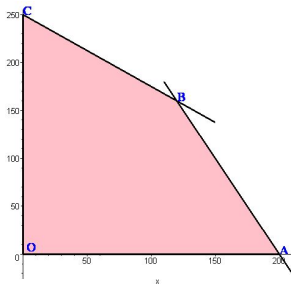
February 2009

Old problem with a new method.

- We recall the problem we solved using the graphic method.

$$\text{Maximize } P = 2x + 1.5y \text{ s.t. } 3x + 4y \leq 1000, 6x + 3y \leq 1200, x, y$$

- This was the sketch of the feasible region.



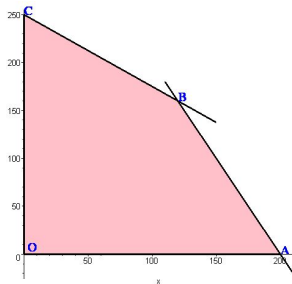
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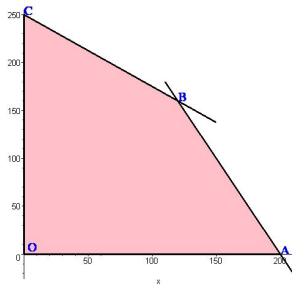
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Example continued.

- We begin by setting up a problem table which will become useful later.

$$\begin{array}{cc|c} x & y & RHS \\ 3 & 4 & 1000 \\ 6 & 3 & 1200 \\ 2 & 1.5 & \end{array}$$

- Note that $x, y \geq 0$ is not listed, since it is always assumed. We are listing inequalities which are always assumed to be of the form \leq with the last row giving the function coefficients.
- We next convert these to equations and write a proper augmented matrix called a simplex tableaux.

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Simplex tableaux.

- We introduce new variables called slack variables so that an inequality $3x + 4y \leq 1000$ is replaced by $3x + 4y + u = 1000$ with the understanding that $u \geq 0$.
Similarly, the second inequality becomes $6x + 3y + v = 1200$ with $v \geq 0$.
- The maximized function declaration is rewritten as $-2x - 1.5y + P = 0$. **Note the changes of signs.**
- This sets up an initial tableaux:

x	y	u	v	P	RHS
3	4	1	0	0	1000
6	3	0	1	0	1200
-2	-1.5	0	0	1	0

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Basic Solution.

- The simplex tableaux always has a set of unit columns (columns of the Identity matrix) which exactly fill up an identity matrix after possible rearrangement.
- The corresponding variables on top of these columns are said to be basic variables and form a basis for the current tableaux.

The number of basic variables must be the same as the total number of rows.

- Corresponding to the basis, we have a **basic solution** to the current tableaux. It is obtained by setting the **non basic variables** to zero value and reading off the solutions of the basic variables from all the equations.
- Thus, for our first tableaux above, the basis is u, v, P and the basic solution is:

$$(x, y, u, v, P) = (0, 0, 1000, 1200, 0)$$

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Modification of the Simplex tableaux.

- Note that our original variables (x, y) have values $(0, 0)$ and thus, this corresponds to the corner point O of our original feasible region.
- If we solve the last equation for its basic variable P , then we have:

$$P = 2x + 1.5y$$

and this says that increasing the values of x or y will increase the value of P .

Note that this is signified by the negative numbers $-2, -1.5$ in the tableaux.

- So we try to make one of them, say x as a basic variable. This means the x column has to become a unit column. Where shall it have its 1?

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Choosing a Pivot.

- You may try and make the pivot at any available non zero entry in the x column. However, some choices will produce negative entries in the last column. This would mean that the new basic solution will have negative values for some basic variables. This is not allowed, since by our set up all the variables have non negative values.

- There is a simple test which will guarantee that we won't get in trouble.

Check the ratios obtained by dividing **non zero entries** in the x -column into the corresponding entries in RHS and choosing the one which gives a minimum value.

- In our example, the choices are $\frac{1000}{3}$ and $\frac{1200}{6} = 200$ respectively. Note that we do not use negative entry -2 . The smallest ratio is 200 in the second row.

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The Cleanup Row Operations.

- Thus, we use the $(2, 1)$ entry 6 to clean out all entries above and below.
- The row operations $R_3 - \frac{-2}{6}R_2$, $R_1 - \frac{3}{6}R_2$ and finally $\frac{1}{6}R_2$ produce the following new tableaux.

x	y	u	v	P	RHS
0	$5/2$	1	$-1/2$	0	400
1	$1/2$	0	$1/6$	0	200
0	$-1/2$	0	$1/3$	1	400

- Note that the new basis is now x, u, P and hence the new basic solution is:

$$(x, y, u, v, P) = (200, 0, 400, 0, 400).$$

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Continued Pivot Operations.

- We note that in terms of x, y variables we are at a point $(200, 0)$ which was our point A . Thus we have marched from O to A on our feasible graph. We also have improved our function value from 0 to 400.
- We notice the negative number $-\frac{1}{2}$ in the function row and this tells us that it would help to put y in the set of basic variables. Explicitly, the last equation is $P = \frac{y}{2} - \frac{y}{3} + 400$, so y wants to get bigger!
- We check the pivot ratios for the second (y) column:

$$\frac{400}{5/2} = 160 \text{ and } \frac{200}{1/2} = 400.$$

Thus the new pivot must be at $(1, 1)$ position.

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Cleanup and end.

- The pivot operations are $R_2 - \frac{1}{5}R_1$ and $R_3 + \frac{1}{5}R_1$ and produce:

x	y	u	v	P	RHS
0	1	$2/5$	$-1/5$	0	160
1	0	$-1/5$	$\frac{4}{15}$	0	120
0	0	$1/5$	$\frac{7}{30}$	1	480

- The new basic is clearly x, y, P and the new basic solution is:

$$(x, y, u, v, P) = (120, 160, 0, 0, 480).$$

Note that this matches our point B in the graph of feasible points.

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Conclusion.

- Note that the last equation is now $P = -\frac{u}{5} - \frac{7v}{30} + 480$ so the non basic variables u, v will not increase its value if they enter the basis.

This is visible since the last row has only non negative coefficients under the variable columns.

So, we are done!!!

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The Simplex Algorithm.

- We only consider a simplified set up outlined in the book. Further complications are reserved for higher level courses.
- Thus, we assume that the starting tableaux has a basis of slack variables and the function P with the RHS entries all non negative. This, in turn means that our maximization problem has all inequalities of the form \leq , except for the assumption that all variables are non negative.
- If there are no negative entries in the last equation (function row), then we are done and the current basic solution is the optimal solution. Our basis always has the function variable P .
- If there is some negative entry in the function row then we call its column the pivot column.
We turn it into a unit column as described below.

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Algorithm Continued.

- Compute the pivot ratios, i.e. the ratios obtained by dividing positive entries in the pivot column into the corresponding RHS entries.
- Pick the smallest among the ratios and pick the corresponding entry in the pivot column as the chosen pivot.
- Do the standard row operations to make the chosen pivot 1 and all other entries in the pivot column zero.
- This has the effect of entering the pivot column variable into the basis. The old basic variable which had its pivot in the current pivot row gets kicked out of the basis.
- Inspect the new tableau for negative entries in the function row and continue.

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What can go wrong?

- The unbounded case. Several problems can come up during the above process. The first is that in the pivot column, we may not have any positive entry to pick as a new pivot. In this case, it can be shown that our problem is unsolvable because the feasible region is unbounded and our function values can grow indefinitely.
- It may happen that the chosen pivot has a pivot ratio 0. In this case, the basis exchange does not increase the function value. This may continue for several steps so that we may get into a loop of basis choices. This does not happen for a small number of variables and is always accompanied by existence of multiple pivot choices. It can be shown that if we make random choices among multiple pivot choices, then the looping can be escaped.

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