Lecture on sections 1.1,1.2

Ma 162 Spring 2010

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Basic Definitions.

- A linear function of one variable x is a function $f(x) = mx + c$ where m, c are constants. Its graph is a straight line, hence it is called linear.
  Example: $f(x) = 3x + 4$.
- A linear function of two variables x, y is of the form $f(x, y) = ax + by + c$. Its graph is a plane in three space.
  Example: $f(x, y) = 3x + 4y + 5$.
- A natural generalization is a linear function of n variables $f(x_1, x_2, \cdots, x_n) = a_1x_1 + a_2x_2 + \cdots a_nx_n + b$ where $a_1, a_2, \cdots, a_n, b$ are constants.
  Example: $f(x, y, z) = 3x + 4y - 5z + 7$.
- These functions are useful in many applications.
- What are examples of functions which are **not** linear?
  Example:
  $f(x) = x^2 + 3x, g(x, y) = x^3 - y^3, h(x, y, z) = xy + yz + zx.$

Real Life functions

Here are some examples of real life functions which behave like linear functions.

- **Distance travelled** If the time interval is short or if an object is moving without acceleration, then $s = at + b$ describes the distance travelled at time t. The coefficient a is the constant **velocity.** Its sign describes if the object is moving away or coming closer.
- **Revenue, Cost and Profit Function.** If x is the number of units sold or manufactured, then we have three natural functions associated with it.
- The cost function is $C(x) = cx + f$ where c is the production cost per unit and f is the fixed cost.
- The revenue function is $R(x) = px$ where p is the price per unit.
- The profit function: $P(x) = R(x) - C(x)$ or $(p - c)x - f$.

Lines or the linear functions of one variable.

We now review how to study the properties of a linear function of one variable x by known geometric properties of its graph, the line.

- **Plane coordinates** Recall that points in the plane are pairs of numbers (x, y), these are the x and y coordinates respectively.
  A point named P with coordinates $(2, 3)$ can be denoted as $P(2, 3)$.
- **Distance Formula.** Recall that the distance between two points $P(a_1, b_1)$ and $Q(a_2, b_2)$ is given by the formula

$$d(P, Q) = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}.$$

  Example. The distance between $P(2, 3)$, $Q(-1, 7)$ is:

$$d(P, Q) = \sqrt{(-1 - 2)^2 + (7 - 3)^2} = \sqrt{9 + 16} = 5.$$

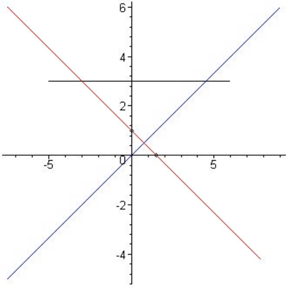
Graph of a function.

The graph of a function $y = f(x)$ consists of all points $P(x, y)$ for which $y = f(x)$.

Example. The graphs of the lines

$$y = \frac{2x}{3}, y = 1 - \frac{2x}{3}, \text{ and } y = 3$$

are shown on the same axes below.



More about lines

- As the graph of a linear function, a line has the equation $y = mx + c$.
- A vertical line does not appear as the graph of linear function. Indeed, it cannot be the graph of **any function.** A vertical line is described by an equation of the form $x = p$ where p is a constant.
- We may combine these two cases and say that the general equation of a line in the plane is of the form:

$$ax + by + c = 0$$

where at least one of a, b is non zero.

Recognizing a Line.

1. Given a line $ax + by + c = 0$, if $b \neq 0$, then it is not vertical. <Indeed,> we can rewrite it as
$$by = -ax - c \text{ or } y = \frac{-a}{b}x + \frac{-c}{b}.$$
Example.
The line $2x - 3y + 5 = 0$ is rewritten as $y = \frac{-2}{-3}x + \frac{-5}{-3}$ or $y = \frac{2}{3}x + \frac{5}{3}$.
2. The slope of the general line $ax + by + c = 0$ is $m = -\frac{a}{b}$ when $b \neq 0$.
If $b = 0$ then the line is vertical and slope is infinite. We sometimes allow an equation $m = \infty$ to express this idea.
3. Positive slope describes a line rising to the right, negative slope describes a line falling to the right, zero slope gives a horizontal line.

Equations of Lines.

- Given two distinct points in the plane, there is a unique line joining them.
  If the two points are (a_1, b_1) and (a_2, b_2), then the slope of this line is:
  $$m = \frac{b_2 - b_1}{a_2 - a_1}.$$
  Example: The slope of the line joining $(2, 3)$ and $(-1, 6)$ is:

  Formula for Slope $$\frac{6 - 3}{-1 - 2} = -1.$$

More Equations of Lines.

- The equation of a line with a given slope m and passing through a point (a, b) is:

  Point Slope Formula. $$y - b = m(x - a)$$

  Example: The slope of the line joining $(2, 3)$ and $(-1, 6)$ is already calculated to be -1. Using it and the point $(2, 3)$, we get:

  $$y - 3 = -1(x - 2) \text{ simplifies to } y = -x + 5 \text{ or } x + y - 5 = 0.$$

- As expected, all formulas need a special handling when the slope becomes infinite, i.e. when the line is vertical. The reader should make this adjustment as necessary.

Intercepts of Lines.

- The intercept of a line refers to its intersection with an axis, when defined.
- Thus, the x-intercept of a line $ax + by + c = 0$ is given by its intersection with the x-axis (i.e. $y = 0$) and is clearly equal to $-\frac{c}{a}$.
  When $a = 0$, this is infinite if $c \neq 0$ and undefined when $c = 0$, i.e. when the line is the whole x-axis.
- Similarly, the y-intercept of a line $ax + by + c = 0$ is given by its intersection with the y-axis (i.e. $x = 0$) and is clearly equal to $-\frac{c}{b}$.
  When $b = 0$, this is infinite if $c \neq 0$ and undefined when $c = 0$, i.e. when the line is the whole y-axis.
- Note that for the line equation $y = mx + c$, the y-intercept is c.

Intercepts form of Lines.

- **Example:** What are the x and y intercepts of $2x - 3y + 5 = 0$?
  Answer: The x-intercept is $-\frac{5}{2}$ and the y-intercept is $\frac{5}{3}$.
- Sometimes, it is desirable to get the equation of a line with given x-intercept p and y-intercept q.
  A nice formula is:

  Intercept Form. $$\frac{x}{p} + \frac{y}{q} = 1$$

  Example: What is the equations of a line with x-intercept -3 and y-intercept 2? **Answer:**

  $$\frac{x}{-3} + \frac{y}{2} = 1 \text{ which simplifies to } 2x - 3y + 6 = 0.$$
- As before, special cases when p or q are zero must be handled separately. These correspond to lines through the origin and it is best to use alternate formulas.