Lecture on sections 2.1, 2.2

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Chapter 2.1, 2.2 Solving Linear Equations.

- As we have seen before, several practical problems lead to formulation and solution of linear equations. First we see several examples of this (compare exercises in Chap. 2.1.)
- Production constraints: Suppose we have a 500 acre farm. Suppose that the production of corn needs \$42 per acre while wheat requires \$30 (EX. 2.1.15).
  Suppose we have $\$18,600$ available and we decide to plant in x acres of corn and y acres of wheat.
  What is the mathematical setup of the problem?
- The constraint due to the farm size is:

$$x + y = 500.$$

The constraint due to the available money is:

$$42x + 30y = 18600.$$

Samples continued.

- Solving the two equations

$$x + y = 500 \text{ and } 42x + 30y = 18600$$

together gives the answer:

$$x = 300, \;\; y = 200.$$

- Thus we **report the answer:** plant 300 acres of corn and 200 acres of wheat!

Investment Example.

- Investment: Two investments yield 8 and 10 percent annually. If a total of \$30,000 is invested and yield is \$2640 per year, how is the fund split between the two investments? (Compare Chap. 2.1.18)
- We assume that amounts in the two funds (with yields of 8% and 10%) are x and y dollars respectively. Then we get the two constraints:

Net value matched. $$x + y = 30000$$

Net return matched. $$0.08x + 0.10y = 2640$$

Answer: Solution gives $x = 18,000$ and $y = 12,000$.

Diet Example.

- Three kinds of food products give different percentages of proteins, carbohydrates and iron.
- Name the three products A, B, C. Then we are given the contents per ounce of the products.
  - Food product A provides 10% of protein, 10% of carbohydrates and 5% of iron needed daily.
  - Food product B provides 6% of protein, 12% of carbohydrates and 4% of iron needed daily.
  - Food product C provides 8% of protein, 6% of carbohydrates and 12% of iron needed daily.

  Calculate how much of each type of food should be eaten to get the 100% RDA of each food. (Compare Chap. 2.1.25).

Diet Example continued.

- Set x, y, z respectively the number of ounces of each of the three products.
- We get three equations:

Protein requirement. $$10x + 6y + 8z = 100$$

Carbohydrate requirement. $$10x + 12y + 6z = 100$$

Iron requirement. $$5x + 4y + 12z = 100$$

- Solution yields:
$$x = 4, y = 2, z = 6.$$
- So we recommend that we eat 4 ounces of product A, 2 ounces of product B and 6 ounces of product C.

Real Life Situations.

- Point to note: In real life problems, we shall find that **we don't get or want** exact equations, but rather inequalities. Also, there is usually some payoff function we are trying to maximize (or some net cost function we are trying to minimize).
- This is often accomplished by converting the inequalities to equations by assigning variable names to the difference between the two sides of the inequalities.
  We will get many possible solutions and develop a method to optimize our payoff (or cost) function. This is the topic of Simplex algorithm coming up later.

The plan of action.

- We note that we know everything about how to solve a system of equations in a single variable.
  We also have learnt several techniques to solve two (or more) equations in two variables.
- For several equations in several variables, the idea is to extend the old methods to **systematically eliminate** one variable at a time and get down to a single variable equation.
- We begin by an example of manipulating equations and then switch over to a convenient method of manipulating the matrix of coefficients, thus avoiding the unnecessary repetition of variable names.
- This is the method of Gauss-Jordan elimination.

Simple Example.

- Let us redo the old example of intersecting lines with the new viewpoint.
  Example: Solve

$$E1: \quad 3x - y = 5 \text{ and } E2: \quad 2x + 3y = 7.$$

- We shall form the matrix of its coefficients with the variables mentioned on the top.

$$\begin{array}{c} \begin{array}{cc|c} x & y & RHS \end{array} \\ \left[\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & 3 & 7 \end{array}\right] \end{array}$$

- Here RHS stands for the right hand side and the vertical bar denotes the equality signs.
- We wish to get rid of one of the variables.

Comtinued Example.

- As a good practice, we eliminate the first one in order, namely x.
- A little thought says that $E3 = E2 - \frac{2}{3}E1$ will give us a new equation $E3: \quad \frac{11}{3}y = \frac{11}{3}$.
  We make a matrix for the new system of $E1, E3$.
- $$\left[\begin{array}{cr|l} x & y & RHS \\ 3 & -1 & 5 \\ 2 & 3 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{cr|l} x & y & RHS \\ 3 & -1 & 5 \\ 0 & \frac{11}{3} & \frac{11}{3} \end{array}\right]$$
- Clearly we can divide the bottom equation by $\frac{11}{3}$ and get ourselves a new matrix:
- $$\left[\begin{array}{cr|l} x & y & RHS \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

Back substitution.

- The current form of our coefficient matrix is said to be in **Row Echelon Form** or **REF** for short.
- It is well suited to solve the last equation for y and then using that answer, to solve the first for x.
- Typically, we now go back to writing out the current equations:
$$3x - y = 5, y = 1.$$
- Using the y-value from the second equation, we deduce $3x - 1 = 5$ or $3x = 6$, i.e. $x = 2$.
This work is called **back substitution.**

A Bigger Example.

- **Compare 2.2.4.** Start with

  $$E1: \ 3x_1+2x_2=0, \ \ E2: \ x_1-x_2+2x_3=4, \ \ E3: \ 2x_2-3x_3=5.$$

- We at once write the coefficient matrix:

  $$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & RHS \end{array} \\ \left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 1 & -1 & 2 & 4 \\ 0 & 2 & -3 & 5 \end{array}\right] \end{array}$$

- Note that x_1 is present in the first equation and the second.
  It is missing from the third.
  We replace the second equation $E2$ by $E2 - \frac{1}{3}E1$ to get rid of it from $E2$.
  The new matrix is:

Example Continued.

- $$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 1 & -1 & 2 & 4 \\ 0 & 2 & -3 & 5 \end{array}\right] R_2 - \frac{1}{3}R_1 \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 2 & 4 \\ 0 & 2 & -3 & 5 \end{array}\right]$$
- We have recorded this operation as $R_2 - \frac{1}{3}R_1$. This means the second row R_2 is replaced by $R_2 - \frac{1}{3}R_1$.
- Be sure to write the changed row as the first term of the expression, always!
- The first column is now clean! We work on the second column next.

The REF.

- Having cleaned x_1 column, we also fix the first row with the idea that we will use it to solve for x_1 at the end.
- In the second column, x_2 appears in the second and third rows. We use the second row entry to clean out the third row entry.
- The operation used is
$$R_3 - \frac{2}{-\frac{5}{3}} R_2 \text{ or } R_3 + \frac{6}{5} R_2.$$
- The new matrix is:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 2 & 4 \\ 0 & 2 & -3 & 5 \end{array}\right] R_3 + \frac{6}{5} R_2 \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 2 & 4 \\ 0 & 0 & -3/5 & \frac{49}{5} \end{array}\right]$$

Final Answer.

- Our equations are now ready to be solved by back substitution. The current form of the matrix is said to be REF (the Row Echelon Form).
- We solve the third equation for x_3, namely $x_3 = \frac{\frac{49}{5}}{\frac{-3}{5}} = -\frac{49}{3}$.
- Plug in this value in the second equation and solve for x_2.

$$-\frac{5}{3}x_2 + 2\frac{-49}{3} = 4 \text{ or } x_2 = -22.$$

- Use these values in the first equation to solve:

$$3x_1 + 2(-22) = 0 \text{ or } x_1 = \frac{44}{3}.$$

This gives the final answer:

$$x_1 = \frac{44}{3}, x_2 = -22, x_3 = -\frac{49}{3}.$$

Conclusion.

- In general, we can follow the same process to solve any number of equations in many variables.
- In general, our equations will be rearranged such that each equation has a distinguished variable (called its pivot variable) which does not appear in lower equations. There may be some leftover variables called non pivot variables.
- Our final answer will express the pivot variables in terms of the non pivot variables, leaving the non pivot variables free to take any values! They will be called the **free variables**.
- Sometimes, we may wipe out all the variables from an equation. In this case, if the RHS is non zero, then we have an inconsistent equation and hence no solution. If the whole equation becomes $0 = 0$, then we leave it among the last such rows. We study this in detail in the next lecture.