Solving Linear Equations.

Ma 162 Spring 2010

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February 1, 2010

Review.

- We have learnt how to write an augmented matrix from a given set of linear equations.
- We then defined what is meant by REF (Row echelon form) and learnt three elementary operations to help us convert the original augmented matrix into an REF.
- Each non zero row of the final REF has a leading non zero entry called its pivot and the meaning of REF is that the successive pivots belong to successively later columns.
- A more refined form of REF is called RREF (reduced row echelon form or the row-reduced form) when all pivots are 1 and their columns are unit columns (i.e. all other entries in the columns) are zero.

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Example 1.

- **Examples of REF.** Here is an augmented matrix in REF decorated with the names of variables on top.

$$A = \begin{array}{c} \begin{array}{cccc|c} x & y & z & w & RHS \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 22 \\ 0 & 1 & 5 & 3 & 24 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

- The pivots are marked and p.p. is $(1, 2, 4, \infty)$. This is strictly increasing, hence we have REF.
- The pivot variables are marked and they are x, y, w while the unmarked z is free.
- We solve the equations for their pivot variables starting from the bottom and using found answers as we go up.

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Solution of example 1.

- The four equations were:

$$x + 2y - z + 2w = 22, \; y + 5z + 3w = 24, \; w = 7, \; 0 = 0.$$

- The very last equation is $0 = 0$ and we ignore that.
- The third equation gives $w = 7$.
- The second gives or $y = -5z - 3w + 24 = -5z - 21 + 24 = -5z + 3$.
- The first equation gives $x = -2y + z - 2w + 22 = -2(-5z + 3) + z - 2(7) + 22 = 11z + 2$.
- The answer to be reported is:

$$x = 11z + 2, y = -5z + 3, z = \text{free or } z, w = 7.$$

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Example 1 finished.

- The above solution can also be written as:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 11z+2 \\ -5z+3 \\ z \\ 7 \end{pmatrix} = z\begin{pmatrix} 11 \\ -5 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 0 \\ 7 \end{pmatrix}.$$

- This is called the parametric form of the solution. The variable z in the answer above can be replaced by a parameter name of your choice. This variable is free to take any value and any such choice yields a solution of the original system.

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Case of No Solution.

- If an REF leads to a situation where one of the pivots is in the last (RHS) column, then the system has no solution, or is inconsistent.
- For example consider:

$$B = \begin{array}{c} \begin{array}{cccc|c} x & y & z & w & RHS \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 22 \\ 0 & 1 & 5 & 3 & 24 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 3t+10 \end{array}\right] \end{array}.$$

- For which values of t is it consistent (or has at least one solution)?
- **Answer:** $3t + 10$ must be zero, otherwise the last equation is false. So, $t = -10/3$ is the answer.

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Example 2: RREF.

- Consider:

$$M = \begin{array}{c} \begin{array}{ccccc} x & y & z & w & RHS \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 7 \\ 0 & 1 & 5 & 0 & 8 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

- This has the same p.p. $(1, 2, 4)$.
- Moreover all the pivot columns are unit columns, i.e. the pivots are all 1 and unique non zero entries in their columns.
- We solve the equations just as in REF case, but the process is shorter.

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Solution of Example 2.

- The equations are:

$$x - z = 7, y + 5z = 8, w = 9.$$

- The answers are

$$w = 9,\ \ y = -5z + 8,\ \ x = z + 7, z = z\ \ z \text{ is free.}$$

- As before, we may write:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 1 \\ -5 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 0 \\ 9 \end{pmatrix}.$$

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Final Comment.

- Suppose our linear system is consistent, i.e. has at least one solution. As already explained, this means no pivot is on RHS in the REF.
- The system then has infinitely many solutions if and only if there is at least one free variable.
  If there is no free variable, then we have a unique solution.
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