The Linear Programming Begins

Ma 162 Spring 2010

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Inequalities.

We discuss inequalities in two or more variables.

- An inequality in one variable looks like $2x + 3 \leq 5$ and is solved by rearranging it so only the variable appears on the left hand side: $x \leq 1$.
- This can also be done graphically thus:
  Convert it to an equation and solve it. Thus:

$$2x + 3 = 5 \text{ leads to } x = 1.$$

- On the number line, plot the point $x = 1$ and notice that all points to the left of it satisfy the inequality and the ones on the right don't.

Examples continued.

- The interval $(-\infty, 1)$ on the number line looks like:

  We verify test values $x = 0$ and $x = 2$ to decide that this interval consists of the solutions and the other part of the number line does not.
- The set of solutions is said to be **the feasible set** of the inequalities used.
- If we similarly handle another inequality, say $3x + 10 \geq 4$, then the solution to the associated equation $3x + 10 = 4$ is $x = -2$ and the interval $[-2, \infty)$ is deduced as before.

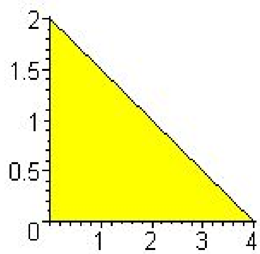
Examples continued.

- If we try to solve both $2x + 3 \leq 5$ and $3x + 10 \geq 4$ together, then we get the intersection of the two intervals. But this can be also explaind thus.
- We solve both associated equations plotting their solutions on the number line.

- By using test points on each interval, say $x = -3, 0, 2$ we pick up the ones which satisfy all the inequalities. This gives the feasible set $[-2, 1]$.

Inequalities in two variables.

- An inequality like $x + 2y \leq 4$ is the next topic. As before, we first convert it to the equation $x + 2y = 4$.
  We note that this is a line and we know how to plot it. It is not difficult to see that the plane is split into two halves so that on one side of the line the inequality is true, while on the other side it is not! Thus, having plotted $x + 2y = 4$, we see that at the origin $O(0, 0)$ the inequality is satisfied. So, we choose as the feasible set the half plane containing the origin.



In the picture, only the first quadrant is shown, since inequalities $x \geq 0, y \geq 0$ are typically going to be part of our conditions.

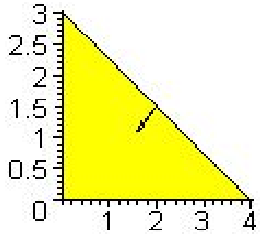
Two variables continued.

- If we have more than one inequalities, then we solve them separately and take the common part. Here is the solution for

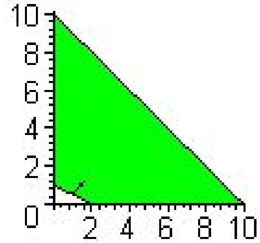
$$x \geq 0, y \geq 0, 3x + 4y \leq 12, x + 2y \geq 2.$$

  As before, the first two inequalities mean we only draw things in the first quadrant. Here are the separate regions for the two inequalities followed by the combined region.

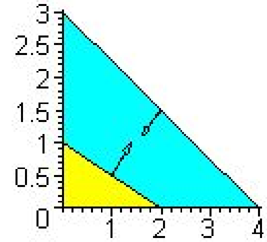
- $3x + 4y \leq 12$



$x + 2y \geq 2$



Combined.



Summary of The Graphical Method.

- On a common graph paper, draw the equations corresponding to each inequality, and mark the regions indicated by each inequality using a directional arrow. In our course, the assumption always puts the region in the first quadrant.
- The directional arrow is usually decided by using a test point. For any inequality at least one of the three points $(0,0),(1,0),(0,1)$ is always a good point to use.
- Take the common part of the plotted regions.
- Calculate and list all the corner points. Make sure that the chosen corner points actually satisfy all ineualities.
- Some corner points, or even some whole lines may be lost, meaning they do not have any points in the region.

Summary Continued.

- The aim of sketching and marking the corner points is to solve an optimization problem for a linear function on the resulting region.
- Review various examples in the section 3.2 where a practical situation leads to a set of inequalities and a linear function to be optimized (i.e. maximized or minimized).
- The first step is to clearly name the variables and write down the inequalities and the function to be optimized.
- Then you sketch the region carefully, provided that you have only two variables. More variables are handled in the next chapter using the Simplex algorithm.
- Then you list all the corner points of the region in order and evaluate the function at each of the corners. The optimum value of the function is among the values at the corner points, with one exception, which we discuss next.

Further Comments.

- If the region is not bounded, then the function may not have a maximum or a minimum. This can be decided by checking along the edges of the polygon running off to infinity.
- **Why does the graphical method work?** We give a brief explanation below.
- **Parametric lines.** Consider a line in the plane, say, $y = 3x + 5$. It passes through a point $(1, 8)$. We want to study the line near this point. So we take a point on this line whose x-coordinate is $1 + t$ and notice that its corresponding y-coordinate shall be $y = 3(1 + t) + 5 = 8 + 3t$.
- In fact, all points of this line can now be described by the parametric equations:

  $$x = 1 + t, y = 8 + 3t.$$

  This is called **the parametric form** of the line. It is useful to be able to calculate such a form near any convenient point!

Explanation Continued.

- Thus for the same line, we could also have started with a point $(-2, -1)$ and concluded a different parametric form

  $$x = -2 + t, y = -1 + 3t.$$

- **Functions on Parametric lines.** Consider a function, say $f(x, y) = 3x + 4y$. We can analyze how it behaves on our parametric line by plugging in the parametric form. Thus we have $f(x, y) = 35 + 15t$.
- This shows that as t increases, so does the function value. Remember that the parameter t was the change in the x-coordinate from the point $(1, 8)$. Thus, as we let x-coordinate increase on our line, the function value increases.

Explanation continued.

- **Conclusion.** Thus on a line in parametric form, a linear function increases or decreases with the parameter depending on the coefficient of the parameter.
- For the above line, if we consider a different function, say $g(x, y) = 3x - y + 2$, then we see that $g(1 + t, 8 + 3t) = 3(1 + t) - (8 + 3t) + 2$ and this simplifies to $g(1 + t, 8 + 3t) = -3$.
- Thus, a linear function on a line is either constant at all points or increases in one of the two available directions and decreases in the opposite direction.

Why Corners?

- Consider the plane region of feasible points that we can plot for our problem. Where would a linear function become maximum on such a region? If we take any point in the interior of our region, then we can draw a little line segment through the point which is still entirely in the region.
- Now if our function is not constant on the line, it would be increasing in one of the two directions and thus would not be maximum at our given point. If by luck, we had chosen a line segment on which the function happens to be constant, we can choose a different segment through the same point and make the same argument. We could not have the same function constant on the second segment as well, for it is clear that then the function would be identically constant on the whole plane Think why!! and our problem has a trivial answer: every point is a maximum point!

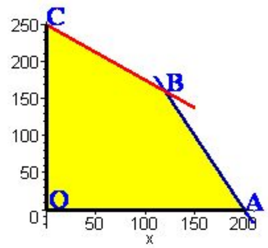
Continued discussion.

- Thus, our maximum point has to be on the boundary!
- It could be on a boundary segment or at a corner. Note that if it is on a segment, but not at a corner, then the function has to be constant on the whole boundary segment (for otherwise we get a contradiction as above).
- This is why it is enough to only check the corner points for locating a maximum.
- This also explains that if we find two maximum points which are corners, then the line joining them must form a boundary line, i.e. they must be adjacent points on the boundary polygon.
- If the region is unbounded, then it has boundary lines running off to infinity and we may find that the maximum point may not exist in the sense that it has to be a point of infinity on one such boundary line.

Some Sample Problems.

- **Problem 3.2.2 by graphing** The problem is to maximize the profit $P = 2x + 1.5y$ subject to $x \geq 0, y \geq 0, 3x + 4y \leq 1000$ and $6x + 3y \leq 1200$
- We first sketch the lines and find their common point. Then we decide on the region.
- Note that in the picture below, the inequality $3x + 4y \leq 1000$ corresponds to the line BC and the inequality $6x + 3y \leq 1200$ matches the line AB. Their regions both point towards the origin, since the origin satisfies both of them! The axes are automatically included with regions pointing towards the first quadrant.

Problem continued.

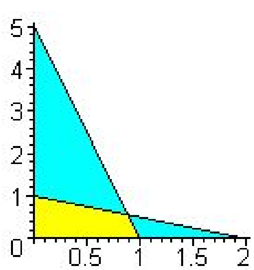
- 

The corners are $O(0,0)$, $A(200,0)$, $B(120,160)$, $C(0,250)$. The values of $2x + 1.5y$ at these are $(0, 400, 480, 375)$.
- So the maximum is at $B(120,160)$ with maximum value 480.

More Solved Problems.

- Consider the problem (similar to B2.8). Suppose that $x + 2y \leq 2$ and $y + 5x \leq 5$ together with $x \geq 0, y \geq 0$. The maximum value of the function $6x + 9y + 2$ on the resulting region occurs $x = \cdots$ and $y = \cdots$. The maximum value of the function is $\cdots$.
- We first convert all inequalities to equations and plot after finding common points. The equations are:

$$x + 2y = 2, y + 5x = 5, x = 0, y = 0.$$

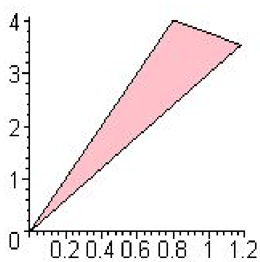
- 

The corners are $(1,0), (8/9, 5/9), (0,1), (0,0)$. The values of $6x + 9y + 2$ at these are $(8, 37/3, 11, 2)$.
- So the maximum is at $(8/9, 5/9)$ and the maximum value is $37/3$.

More Solved Problems.

- Consider the problem (similar to B2.9). Suppose that $y \leq 5x, y \geq 3x$ and $x/4 + y/5 \leq 1$ together with $x \geq 0, y \geq 0$.
  The maximum value of the function $x + y$ on the resulting region occurs $x = \cdots$ and $y = \cdots$. The maximum value of the function is $\cdots$.
- We first convert all inequalities to equations and plot after finding common points. The equations are:

$$y = 5x, y = 3x, x/4 + y/5 = 1, x = 0, y = 0.$$

  Note that the last two do not contribute to the picture!
- 

  The corners are $(0, 0), (4/5, 4), (20/17, 60/17)$.
  The values of $x + y$ at these are $(0, 24/5 = 4.8, 80/17 = 4.7059)$.
- So the maximum is at $(4/5, 4)$ and the maximum value is 4.8.