DEPARTMENT OF MATHEMATICS

Solutions to Ma 162 Second Exam March 3, 2008

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1. Consider the following Matrices and answer the questions.

   In each case, either calculate the expression or explain why it is not defined.

$$A = \begin{bmatrix} -3 & 2 & 3 \\ -4 & -4 & 3 \\ 2 & 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -3 & -5 & 2 \\ 0 & 3 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 5 & -4 \\ 0 & -5 \end{bmatrix} \quad D = \begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix}$$

(a) $D^2 - 2\,D$

$$\begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix} - 2\begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix} =$$
$$\begin{bmatrix} 20 & -8 \\ -8 & 16 \end{bmatrix} - \begin{bmatrix} 4 & -8 \\ -8 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

(b) AB

This is not defined A is 3×3 while B is 2×3

(c) BC

$$\begin{bmatrix} -3\cdot 1 + (-5)\cdot 5 + 2\cdot 0 & -3\cdot 1 + (-4)(-5) + 2(-5) \\ 0\cdot 1 + 3\cdot 5 + 5\cdot 0 & 0\cdot 1 + 3(-4) + 5(-5) \end{bmatrix} = \begin{bmatrix} -28 & 7 \\ 15 & -37 \end{bmatrix}$$

(d) $2\,A + 5\,C$

This is not defined. $2A$ is 3×3 and $5C$ is 3×2. These matrices must be the same size in order to be added.

(e) CD

$$\begin{bmatrix} 1 & 1 \\ 5 & -4 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 26 & -20 \\ 20 & 0 \end{bmatrix}$$

2. (a) Find the inverse of the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - \frac{1}{5}R_1} \left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 0 & \frac{12}{5} & -\frac{1}{5} & 1 \end{array}\right] \xrightarrow{\frac{5}{12}R_2} \left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 0 & 1 & -\frac{1}{12} & \frac{5}{12} \end{array}\right] \xrightarrow{R_1 - 3R_2}$$
$$\left[\begin{array}{cc|cc} 5 & 0 & \frac{5}{4} & -\frac{5}{4} \\ 0 & 1 & -\frac{1}{12} & \frac{5}{12} \end{array}\right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{12} & \frac{5}{12} \end{array}\right]$$

Shortcut: Note that the simple formula for the inverse of a 2×2 matrix will directly give:

$$A^{-1} = \frac{1}{5\cdot 3 - 3\cdot 1}\begin{bmatrix} 3 & -3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3/12 & -3/12 \\ -1/12 & 5/12 \end{bmatrix}$$

Answer: $\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{12} & \frac{5}{12} \end{bmatrix}$

(b) Find the inverse of the matrix $B = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + \frac{2}{3}R_1}$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 1 \end{array}\right] \xrightarrow{3R_3} \left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 \end{array}\right] \xrightarrow{R_2-R_3}$$
$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & -3 \\ 0 & 0 & 1 & 2 & 0 & 3 \end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 3 & 0 & 3 \\ 0 & 1 & 0 & -3 & 1 & -3 \\ 0 & 0 & 1 & 2 & 0 & 3 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 & -3 \\ 0 & 0 & 1 & 2 & 0 & 3 \end{array}\right]$$

The answer is: $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -3 \\ 2 & 0 & 3 \end{bmatrix}$. **There is no easy shortcut for 3×3 or bigger matrices.**

3. Set this problem up, by stating the chosen variables, the function to be maximized and **all** the inequalities. **Do not solve the problem.**

   The juice company "Exotics" has three lines of juice mixes.

   Each carton of blend A contains 10 ounces of Peach concentrate and 6 ounces of Mango paste.

   Each carton of blend B contains 9 ounces of Peach concentrate and 7 ounces of Orange concentrate.

   Each carton of blend C contains 7 ounces of Peach concentrate, 5 ounces of Orange concentrate and 4 ounces of Mango paste.

   The company has 9000 ounces of Peach concentrate, 6000 ounces of Orange concentrate and 12000 ounces of Mango paste in stock.

   If the profits per carton of the blends A, B, C are $1.20, 1.50, 1.50$ dollars respectively, how many cartons of each blend should be produced?

   We define variables and set up a maximal linear programming problem (LPP) associated with this question.

   Let x be the ounces of blend A produced.
   Let y be the ounces of blend B produced.
   Let z be the ounces of blend C produced.
   Then the profit function is $P = 1.2x + 1.5y + 1.5z$. With the variables defined, we will use a total of $10x + 9y + 7z$ ounces of peach concentrate. Since the company has only 9000 ounces of peach concentrate $x, y,$ and z are constrained by:
   $10x + 9y + 7z \leq 9000$.
   Similarly,
   $7y + 5z \leq 6000$,
   $6x + 4z \leq 12000$.
   Don't forget, we also have the restrictions, $x \geq 0, y \geq 0, z \geq 0$.

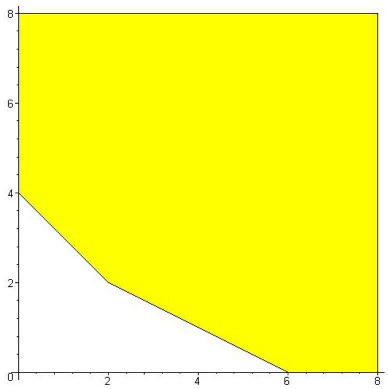
4. i) Sketch and shade the region described by the inequalities. Compute the coordinates of the corner points and mark them on your graph.

   $0 \leq x\ ,\ 0 \leq y$

   $4 \leq x + y$

   $6 \leq x + 2\,y$

   We graph this region below.



To find the corner points we solve the system $4 = x + y$, $6 = x + 2y$. We may solve this system by substitution. The first equation implies $y = 4 - x$. Substitution into the second equation yields

$$6 = x + 2(4 - x) \qquad \text{or} \qquad x = 2.$$

Together $x = 2$, and $y = 4 - x$ implies $y = 2$. All three corners are $(0, 4)$, $(6, 0)$, and $(2, 2)$.
ii) Find the minimum value of the function, $C = 4\,x + 6\,y$ on the region.
Compute the cost at each of the corner points. We get 24 at $(0, 4)$, 24 at $(6, 0)$, and 20 at $(2, 2)$. Thus 20 is the minimum. Note that the cost increases as x and y increase. So that 20 is in fact the minimum despite the fact that our region is unbounded.

5. Here is an intermediate tableau associated with a maximal LPP.

x	y	z	s	t	P	constants
2	3	0	1	0	0	14
-1	1	1	0	-2	0	4
6	-5	0	0	15	1	12

i) Circle the pivot element and carry out the next iteration of the simplex method.

The -5 is the only negative entry in the bottom row, this is in the second column. The ratios associated with this column are $\frac{14}{3}$ and $\frac{4}{1}$. As $4 < \frac{14}{3}$ the pivot is entry of the second column in the second row. To carry out the simplex algorithm we apply the row operations $R_1 - 3R_2$ and $R_3 + 5R_2$. After the next iteration of the simplex method we obtain:

x	y	z	s	t	P	constants
5	0	-3	1	6	0	2
-1	1	1	0	-2	0	4
1	0	5	0	5	1	32

ii) Using your answer in the first part, report the solution to the original maximal LPP.

From this tableau we can read off the maximum, $P = 32$, which is achieved at $(x, y, z) = (0, 4, 0)$.

6. You are given the minimization problem:

Minimize the objective function: $C = 10\,x + 3\,y + 10\,z$

Subject to:

$$20 \le 2\,x + y + 5\,z$$

$$30 \le 4\,x + y + z$$

$x \ge 0, y \ge 0$ and $z \ge 0$

The final tableau **for the dual problem** is:

u	v	x	y	z	P	constants
0	0	2	-9	1	0	3
0	1	$1/2$	-1	0	0	2
1	0	$-1/2$	2	0	0	1
0	0	5	10	0	1	80

Using this give the solution to the primal problem (i.e. original minimal LPP):

The minimal value of C is 80 which occurs at the values $(x, y, z) = (5, 10, 0)$. This is an application of theorem 1 on page 250 of our text.