

DEPARTMENT OF MATHEMATICS

Problems from old third exams. The actual exam is expected to have 8 questions. It is recommended that all the following problems be reviewed along with the homework and quizzes.

Useful formulas as they appear on exam front.

- (1) Simple Interest: $I = Prt$. Accumulation: $A = P(1 + rt)$.
- (2) Compound Interest Accumulation: $A = P(1 + i)^n$. Present value: $P = A(1 + i)^{-n}$.
- (3) Effective rate: $r_{eff} = (1 + \frac{r}{m})^m - 1$.
- Annuity: (4.1) Sum: $S = R \frac{((1+i)^n - 1)}{i}$.
(4.2) Present value: $P = R \frac{(1 - (1+i)^{-n})}{i}$.
- Set counting:
 - (5) Two sets: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
 - (6) Three sets: $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

1. Suppose that A, B and C are sets with 93, 69 and 86 members respectively. Calculate the indicated quantities. Display correct formulas or appropriate Venn diagrams.

i) If $A \cap B$ has 45 members, then $A \cup B$ has members.

Answer: Using the two set counting formula (5), we get:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 93 + 69 - 45 = 117.$$

ii) If it is further known that $A \cap C$ has 56 members, then $A \cup C$ has members.

Answer: Using the two set counting formula(5), we get:

$$n(A \cup C) = n(A) + n(C) - n(A \cap C) = 93 + 86 - 56 = 123.$$

iii) If, in addition $B - C$ has 39 members, then $B \cap C$ has members.

Answer: Note that by the same formula, we get $n(B) = n(B - C) + n(B \cap C)$, since $B - C$ and $B \cap C$ have empty intersection.

Hence, $69 = 39 + n(B \cap C)$ so $n(B \cap C) = 69 - 39 = 30$.

iv) Finally, if we are given that the intersection of all three sets A, B, and C has 16 members, then the union of all three sets has members.

Answer: Use the three set counting formula (6) and the above results to get:

$$n(A \cup B \cup C) = 93 + 69 + 86 - (45 + 56 + 30) + 16 = 133.$$

2. Bill invests \$4400 at 7% compounded quarterly.

a) After 10 years, his investment will grow to \$

Answer: Since this is a single investment compounded, we use formula (2). We have $r = 0.07$, $m = 4$, so $i = 0.07/4$. Also $t = 10$, so $n = 10 \cdot 4 = 40$. Also $P = 4400$.

So the answer is: $4400(1 + 0.07/4)^{40} = 8807.03$.

b) His investment will have grown to \$22,000. after years.

Answer: The same formula now gives:

$$22000 = 4400(1 + 0.07/4)^{4t}$$

where the new number t of years is being asked.

Taking logs and simplifying:

$$\log(22000/4400) = 4t(\log(1 + 0.07/4))$$

and this gives $4t = 92.7703$ or $t = 23.19$.

3. Jack needs \$10,000 after 10 years from now.

How much would he have to invest now at 7% interest compounded annually to have that sum available? Be sure to show the formula used.

Answer. This is still a single investment with compound interest, so we use formula (2) and note that $i = 0.07/1$, $n = 1 \cdot 10 = 10$ and $A = 10000$.

So

$$10000 = P(1 + 0.07)^{10} \text{ or } P = 10000(1 + 0.07)^{-10} = 5083.49.$$

4. You plan to pay for the purchase of a house with a 15 year loan of 120 thousand dollars at 6.6% interest compounded monthly. What is your monthly payment of principal and interest and what is the your total payout?

Monthly Payment: dollars.

Be sure to show the formula used.

Answer: This is a monthly payment situation, so we need formula 4.1 or 4.2. Also, we are given P the present value, so use 4.2.

Using the values $r = 6.6/100$, $m = 12$, $t = 15$, $P = 120000$, we get $i = 6.6/1200$, $n = 12 \cdot 15 = 180$ and the equation:

$$120000 = R(1 - (1 + 6.6/1200)^{-180})/(6.6/1200).$$

Evaluating, we get: $120000 = R(114.0752759)$ or $R = 120000/114.0752759 = 1051.94$.

Total payout means total of all the payments made during the term of the loan. dollars.

This does not need any formula: We pay the above amount $R = 1051.94$ monthly for 15 years, so the payout is $1051.94(12)(15) = 189348.65$.

5. Suppose that you decide to save up for a car purchase in 5 years. You want to have \$14,000 saved, and you will make monthly payments into an account paying 8% interest compounded monthly.

Be sure to show the formulas used.

- a) How much should your monthly payment be? **Answer:** This is again an annuity case, since we are making periodic payments. This time, the future value 14000 is given so use (4.1).

$$14000 = R \left(\frac{(1 + 8/1200)^{12 \cdot 5} - 1}{8/1200} \right)$$

or $14000 = R \cdot 73.47686070$ or $R = 14000/73.47686070 = 190.5361752$, so you report 190.54 as the monthly payment.

- b) If at the end of 5 years, you decide to keep the old car another year and continue making the same payments, how much **additional money** will accumulate into the account?

Answer: The net accumulation after 6 years at the same payment rate would be: 17534.50654 or 17534.51, so we would have an additional 3534.51 dollars extra over the original sum of 14000.

Note that the extra accumulation 3534.51 in the 6th includes a total of $(190.54)12 = 2286.48$ in payments and the remaining 1248.04 come from accumulated interest!

6. In this problem, assume a 360 day year. "Rent a Check" will loan you 55% of your paycheck of \$300 for 7 days. After 7 days, you pay back the loan plus \$10.

a) Calculate the annual simple interest rate for this service. Answer:
per cent.

Surprised? Now you know what a loan shark is!!

Answer: This is a simple interest rate calculation using formula 1. We have $t = 7/360$, $P = 300(0.55)$ and $I = 10$, so

$$r = \frac{I}{Pt} = \frac{10}{165(7/360)} = 3.116883117$$

So, the rate is 311.69%.

b) Across the street, "The Loan Shop" will loan you 5% more of your paycheck and charges you an interest of \$11. Is this a lower rate? Explain your answer by comparing the rates. Be sure to show all work.

Answer: This is the same problem with different P and I .

Thus, the new rate is:

$$r = \frac{I}{Pt} = \frac{11}{(180)(7/360)} = 3.142857143$$

which is 314.29%, or even higher!

7. Sam has a library of different but colorful books. He has 6 red books, 13 blue books, and 4 yellow books.

a) Suppose he lets a friend take home three books, one of each color. How many possibilities are there ? Answer:

Answer: By the multiplication principle, we get the answer as $(6)(13)(4) = 312$

b) Suppose instead he decides his friend can only borrow two books and they must be the same color. How many possibilities now?

Answer: There are three separate cases, depending on the color chosen. The friend can choose 2 red books in $C(6, 2) = \frac{(6)(5)}{2} = 15$ ways, blue books in $\frac{(13)(12)}{2} = 78$ ways, and yellow books in $\frac{(4)(3)}{2} = 6$ ways.

The total count is $15 + 78 + 6 = 99$. **Note** that the final numbers are not multiplied, but simply added. We cannot choose different colors simultaneously!

8. There are 12 members in an executive committee. A slate of three members is to be nominated from them to work as officers: the President, the Secretary and the Treasurer.

a) How many different slates can be formed from the executive committee?

Answer:

Answer: Imagine setting up chairs with titles on them. The president's chair can be filled in 12 ways. Now only 11 members are available for the secretary's chair and then only 10 left for the treasurer's chair.

Since these choices have to be made at the same time to make a slate, the answer is $(12)(11)(10) = 1320$.

We could also explain this by the formula $P(12, 3) = (12)(11)(10)$.

b) Suppose that Mr. Smith is already the chosen nominee to become the Secretary. In that case, how many ways are there to complete the slate ?

Answer:

Answer: Now remove Mr. Smith from available candidate list and note that only two more positions have to be filled. So the answer is $P(11, 2) = (11)(10) = 110$.

9. Jack and Lisa are getting married and hoping to buy a small house worth \$60,000. They need to come up with \$12,000 for down payment and have only \$2,000 saved for this.

- (a) Fortunately, their parents offer them a simple interest loan for the remaining \$10,000 at a very generous 4.5% APR (annual percentage rate) for a period of 5 years. If they accept this offer, how much net amount would they pay back to their parents after the 5 years?

Answer: This is formula 1 applied:

$$A = 10000(1 + 4.5/100(5)) = 12250.$$

The interest expense is thus \$2250

- (b) After discussion with the mortgage company, the aggressive sales manager said that he can do better and offered them an APR of 4.1% compounded annually for the same period and the same amount. “By the way,” he added casually that “there is a small processing fee of only \$40”.

Should Jack and Lisa go for the parents’ offer or the manager’s offer? Justify your answer with solid numbers!

Answer: This time, the net expense is interest plus \$40, but we need to use the compound interest formula 2.

$(10000)(1 + 4.1/100)^5 + 40 = 12225.13 + 40 = 12265.13$. Thus, they lose more with the bank!

- (c) If the manager were to offer an APR of 4%, but now compounded monthly and with the same processing fee, what should Jack and Lisa do? Again justify your answers with precise calculated numbers.

Answer: This time our accumulation comes out 12209.97. Even with \$40 fee, this is \$12249.97 or 3 cents cheaper than the parents!

10. (a) If the manager offers Jack and Lisa an APR of 4% compounded monthly for a period of 15 years, provided they borrow \$70,000, what would their monthly payment be?

Answer: We use formula (4.2) to write:

$$70000 = R \left(\frac{1 - (1 + 4/1200)^{-(12)(15)}}{4/1200} \right)$$

Solution gives $R = \frac{70000}{135.1921388} = 517.78$.

- (b) Jack and Lisa now do some calculations and figure that they can afford between \$500 and \$550 in monthly payments. If they can get an APR of 5% compounded monthly and want to pay off the loan in 10 years, what are the least and most amounts they can hope to borrow?

Answer: We use $R = 500$ and 550 respectively with $r = 0.05$ and $t = 10, m = 12$. Then the values of P from the same formula give: \$47140.68 and \$51854.75 respectively. These are the limits for the 10 year repayment.

- (c) Jack and Lisa want to get new furniture but decide that they want no more loans and they can wait for two years before they get the furniture that they love. They decide to start a savings account which pays an APR of 3.5% compounded monthly and decide to put \$250 into it every month. How much can they afford to pay for the furniture after the two years?

Answer: Here the future value is desired with $R = 250, r = 3.5/100, m = 12, t = 2$. Thus the formula 4.1 gives the answer \$6205.62.

11. Answer the following:

- (a) In a survey of 1250 likely voters in Fayette county, 670 said that they would vote for the mass transit proposal and 875 said they would vote for the marriage amendment. If 50 were not going to vote for either of the two amendments, how many were going to vote for both the amendments?

Answer: Let A and B be the sets of people voting for the mass transit and Marriage amendment respectively.

We know $n(A) = 670$, $n(B) = 875$. Clearly, $n(A \cup B) = 1250 - 50 = 1200$. Thus by formula 5,

$$1200 = 670 + 875 - n(A \cap B)$$

so the desired answer is 345.

In the same survey, how many were going to vote for only one of the two amendments?

Answer: This asks you to add up $n(A - A \cap B)$ and $n(B - A \cap B)$ and these are clearly $670 - 345$, $875 - 345$ respectively. The answer is $325 + 530 = 855$.

This can also be calculated as $n(A \cup B) - n(A \cap B)$.

- (b) In a survey of 400 students it was found that 220 usually went to the student center cafeteria to eat, 190 often used some other nearby campus cafeteria, while 70 claimed that they go to both regularly. The rest reported that they don't eat on campus. How many don't eat on campus?

Answer: Let A be the set of center cafeteria clients and B be the set of other cafeteria students. Then $n(A \cup B)$ is the number of students eating on campus and the rest from 400 eat elsewhere.

We see from formula 5,

$$n(A \cup B) = 220 + 190 - 70 = 340.$$

So the left over $400 - 340 = 60$ eat away from the campus.

- (c) In a survey, 100 voters were asked which of the three election issues -terrorism, economy and morality was important for them. They could vote for more than one choice if desired.

Of these, 10 declared that every issue was very important for them.

There were 40 voters who considered that the terrorism issue was important, 26 thought that the economy was important and 65 thought that the morality issue was very important.

- Determine the number of voters who found that two or more issues were important for them.
- Determine the number of single issue voters, i.e. voters who considered only one of the three issues to be important.

Answer: Let A, B, C be the sets of voters who consider terrorism, economy and morality important respectively.

Here is what we know.

- $n(A \cup B \cup C) = 100$, $n(A \cap B \cap C) = 10$.
- $n(A) = 40$, $n(B) = 26$, $n(C) = 65$.
- From formula 6, we deduce that:

$$100 = 40 + 26 + 65 - (n(A \cap B) + n(B \cap C) + n(C \cap A)) + 10.$$

Thus:

$$n(A \cap B) + n(B \cap C) + n(C \cap A) = 40 + 26 + 65 + 10 - 100 = 41.$$

- The set voters voting for two or more issues is the union of the three sets $A \cap B, B \cap C, C \cap A$. We note that the intersection of any two of these is always the same set $A \cap B \cap C$ which has 10 elements.
- Now, formula 6 applied to these three sets gives:

$$n(A \cap B \cup B \cap C \cup C \cap A) = 41 - (10 + 10 + 10) + 10 = 21.$$

This is the number of people voting for at least two issues!

- Finally, single issue voters are the remaining $100 - 21 = 79$.