DEPARTMENT OF MATHEMATICS

Ma213 - EXAM #2 Fall 2012

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO

There are 7 problems and a total of 6 pages including this one. You may only use the two sheets of formulas distributed by us. It should not have any additional writing on it.

Problem	Maximum Score	Earned Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	20	
7	10	
WHS	10	
Total	100	

NAME: ______________________________

SECTION No: ______________________________

LAST 4 digits of Student No.: ______________________________

1. (20 points) Consider a particle moving in space by the formula:

   Grading Policy: 4 points each. Simplification is not required. Be kind to earlier mistakes causing later ones. Generally, one point per mistake until credit is gone.

   $P(t) = \mathbf{r}(t) =< 2t, -3t, 1 - t^2 >$.

   Answer the following questions.

   (a) Find the formulas for the velocity and acceleration at time t.

   Answer: $v =< 2, -3, -2t >, a =< 0, 0, -2 >$

   (b) Find formulas for the unit tangent **T**, unit normal **N** and **B**.

   Answer: $T = \frac{1}{\sqrt{13+4t^2}} < 2, -3, -2t >, B = \frac{1}{\sqrt{52}} < 6, 4, 0 >, N = \frac{1}{2\sqrt{169+52t^2}} < -8t, 12t, -26 >$.

   (c) Find the equation of the tangent line to the path at time $t = 0$.

   Answer: $\mathbf{r}(t) =< 0, 0, 1 > + < 2, -3, 0 > t$.

   (d) Determine the curvature of the path in space at time $t = 0$.

   Answer: $\frac{\sqrt{52}}{\sqrt{13}^3}$.

   (e) Is the path contained in a plane? Justify your answer.

   Answer: Yes, because $\frac{1}{|P(t) \times v(t)|} P(t) \times v(t) =< \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, 0 >$ is a constant vector.

   Or, because B is constant.

   An even simpler reason is that $3x + 2y = 0$ is satisfied by our curve!

2. (10 points)

Grading Policy: Points 3+3+4. Proof by $\epsilon - \delta$ is good, but not needed. The wording by students can be sloppy; charge a point for possibly wrong statements.

Answer the following questions.

(a) Describe the domain of the function

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 1}}.$$

Answer: Points (x, y, z) on the **outside** of the sphere $x^2 + y^2 + z^2 = 1$.

(b) Describe the domain of the function

$$g(x, y) = \log\left(\frac{1}{x + y}\right).$$

Answer: Points (x, y) for which $x + y > 0$, i.e. points above the line $x + y = 0$.

(c) Determine if the following limit exists and if it does, then find its value. **It is necessary to show your argument. Just an answer will not earn any credit.**

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2 + y^3}{x^2 + y^2}.$$

Answer: In polar coordinates, we have

$$\frac{r^3(\cos^3(\theta) + \cos(\theta)\sin^2(\theta) + \sin^3(\theta))}{r^2} = r(\cos^3(\theta) + \cos(\theta)\sin^2(\theta) + \sin^3(\theta)).$$

Now since the first factor r has limit 0 and the second factor is bounded (say, by 3) the limit of the product is 0.

3. (10 points)

Consider the functions:

$$f(x,y) = x^3 + xy - 3y^2, g(u,v) = u^2 - v^2, h(u,v) = \frac{1}{u-2v}.$$

Calculate the indicated quantities.

(a) Calculate $\nabla(f), \nabla(g), \nabla(h)$.

Grading Policy: 3 points - one each.

Answer: $< 3\,x^2 + y, x - 6\,y >, < 2u, -2v >, < -\frac{1}{(u-2v)^2}, \frac{2}{(u-2v)^2} >$.

(b) let $z = f(x,y), x = g(u,v), y = h(u,v)$. Using the above calculations (or otherwise), calculate $\frac{\partial z}{\partial v}$.

(Here, you may leave the expression unsimplified.)

Grading Policy: 2+2 points. Final answer should be simplified for full credit.

Answer: $\frac{\partial z}{\partial v} = (3\,x^2 + y)(-2v) + (x - 6\,y)(\frac{2}{(u-2v)^2})$.

Further, find the value of this derivative at $u = 1, v = -1$. Here you must simplify your final answer.

Answer: Evaluates to 2/9.

(c) Suppose that x, y, z are related by the equation

$$x^3 + y^3 + z^3 - 5xyz = 6.$$

Calculate $\frac{\partial z}{\partial y}(1, 1, -1)$.

Grading Policy: 3 points.

Answer: $\nabla(f) =< 3x^2 - 5yz, 3y^2 - 5xz, 3z^2 - 5xy >$ and this evaluates to $< 8, 8, -2 >$. This gives: $\frac{8}{2} = 4$.

4. (10 points) The cost to make a strong box of length x, width y and height z is known to be $f(x, y, z) = 36xy + 12yz + 12xz + 10$ dollars. Answer the following questions.

   (a) What is the cost of a box with length 10, width 8 and height 7?

   Grading Policy: 2 points.

   Answer: 4402$.

   (b) Find the linear approximation to $f(x, y, z)$ at $(x, y, z) = (10, 8, 7)$.

   Grading Policy: 4 points, 3 for $\nabla(f)$ and 1 for using the correct formula. Charge 1 point for inconsistent function value (from part (a)).

   Answer: $\nabla(f) =< 372, 444, 216 >$ at the point.

   So, $L(x, y, z) = 4402 + 372\Delta(x) + 444\Delta(y) + 216\Delta(z)$.

   (c) Using the linear approximation, estimate the cost of a box whose length and height are increased by 0.5 and width is reduced by 0.1. Final answer should be a correct decimal number.

   Grading Policy: 3 points. Again, charge for inconsistency.

   Answer: $4402 + 249.6$.

5. (10 points) Let F be the surface in three space given by

$$x^3 + 2xy + y^2 - 3yz + z^2 = 2.$$

   (a) Calculate the equation of the tangent plane to the surface F at the point $(1, 1, 1)$.

   Grading Policy: 3 points for correct gradient and 2 points for building the correct equation.

   Answer: Tangent plane: $5(x - 1) + 1(y - 1) - 1(z - 1) = 0$.

   (b) Use this equation to estimate c, if $(1.2, 0.9, c)$ is on the surface.

   Grading Policy: 3 points for correct equation, 2 points to solve.

   Answer: c satisfies $5(1.2 - 1) + (0.9 - 1) - (c - 1) = 0$, so $c = 1.9$.

6. (20 points) Let $f(x, y, z) = x^2 + xy + yz$.

Grading Policy: Points 7+7+2+2+2.

(a) Calculate $\nabla(f)(1, 2, 5)$.

Grading Policy: gradient 4 and evaluation 3.

Answer: Gradient $= < 2x + y, x + z, y >$ evaluates to $< 4, 6, 2 >$.

(b) Calculate the directional derivative of f in the direction of $< 1, -2, 3 >$ at the point the point $(1, 2, 5)$ (In notation: $D_v(f)(1, 2, 5)$).

Grading Policy: Gradient 4, evaluation of formula 3. One off for forgetting to divide by length.

Answer: The derivative is $\frac{-2}{\sqrt{14}}$.

(c) Find a vector v such that $D_v(f)(1, 2, 5) = 0$.

Answer: Perpendicular to the gradient: Many possible answers. For example $< 6, -4, 0 >$.**corrected typo.**

(d) Find a vector w such that $D_w(f)(1, 2, 5)$ has largest value. What is this largest value?

Answer: $w =< 4, 6, 2 >$ and the derivative is $\sqrt{4^2 + 6^2 + 2^2} = 2\sqrt{14}$.

(e) Find a vector p such that $D_p(f)(1, 2, 5)$ has smallest value. What is this smallest value?

Answer: $w = - < 4, 6, 2 >$ and the derivative is $-\sqrt{4^2 + 6^2 + 2^2} = -2\sqrt{14}$.

7. (10 points)

Grading Policy: Points 4+6. Half for gradient, half for solutions.

(a) Calculate all the critical points of $f(x, y) = x^2 + y^2 + xy + x - 2\,y + 5$.

Answer: We solve

$$[2\,x + y + 1, 2\,y + x - 2] = [0, 0].$$

Solution is $[x = -4/3, y = 5/3]$.

(b) Calculate all the critical points of $g(x, y, z) = x^2 + y^2 - z^2 + x - y + z + yz$.

Answer: We solve

$$[2\,x + 1, 2\,y - 1 + z, -2\,z + 1 + y] = [0, 0, 0].$$

Solution is: $[x = -1/2, y = 1/5, z = 3/5]$.