DEPARTMENT OF MATHEMATICS

Ma213 - FINAL EXAM Spring 2013 **Grading Policy:** Answers Provided.

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO

There are 9 problems and a total of 10 pages including this one. **You are permitted to use:** two sheets of your own notes (front and back). **These will not be shared.**
Calculators are permitted, so long as they are not capable of wireless communication.

Problem	**Maximum Score**	**Earned Score**
1	20	
2	15	
3	15	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
WHS	10	
Total	120	

NAME: ______________________________

SECTION No: ______________________________

LAST 4 digits of Student No.: ______________________________

1. 20 points

   Grading Policy: 4 points each. Take 1 point off for each mistake, until credit is exhausted.

   (a) Find all vectors $w =< a, b, c >$ which are perpendicular to both

   $$v_1 =< 2, -2, 3 >, v_2 =< 3, 1, -7 >$$

   and such that $c^2 = 64$.

   Answer: $v_1 \times v_2 =< 11, 23, 8 >$. We need $c = \pm 8$. So the answer is $w = \pm < 11, 23, 8 >$.

   (b) Find the equation of a plane through the point $P(2, 3, 5)$ and parallel to both $v_1 =< 2, -2, 3 >$ and $v_2 =< 3, 1, -7 >$

   Answer: The plane has normal $< 11, 23, 8 >$, so equation $11(x-2)+23(y-3)+8(z-5) = 0$.

   (c) Find the equation of the plane passing through $P(2, 3, 5)$, $Q(4, 1, 8)$ and $R(5, 4, -2)$.

   Answer: The plane contains vectors $\overrightarrow{PQ} =< 2, -2, 3 >$ and $\overrightarrow{PR} < 3, 1, -7 >$, so the answer is same as above $11(x - 2) + 23(y - 3) + 8(z - 5) = 0$.

   (d) What is the area of the triangle formed by $P(2, 3, 5)$, $Q(4, 1, 8)$ and $R(5, 4, -2)$?

   Answer: Answer is $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}| < 11, 23, 8 > | = \frac{1}{2}\sqrt{714} = 13.36038922$.

   (e) Find the equation(s) of the line joining $Q(3, 1, 10)$ and $R(5, 5, 4)$. What is the midpoint of the line segment QR?

   Answer: The parametric form is $r(t) = (1 - t) < 3, 1, 10 > +t < 5, 5, 4 >$. Symmetric equations are:

   $$\frac{x-3}{2} = \frac{y-1}{4} = \frac{z-10}{-6}.$$

   Midpoint is $\frac{Q+R}{2} = (4, 3, 7)$.

2. 15 points

Grading Policy: $5 + 5 + 5 = 15$ points. Charge 2 points for each serious mistake. Only one point for minor mistakes.Do not charge points if correct integration is carried out without details of substitution.

Consider the space curve C given by the parametric form:

$$\mathbf{r}(t) =< \sin(2\,t)\,, -t, \cos(2\,t) > .$$

Answer the following questions.

(a) Determine the unit tangent $\mathbf{T}$, and the unit bitangent $\mathbf{B}$ to the curve C at $t = \pi$.

Answer: $v =< 2\,\cos(2\,t)\,, -1, -2\,\sin(2\,t) >, a =< -4\,\sin(2\,t)\,, 0, -4\,\cos(2\,t) >$.

So,

$$T = \frac{1}{\sqrt{5}} v = \frac{1}{\sqrt{5}} < 2, -1, 0 > .$$

$v \times a =< 4\cos(2t), 8, -4\sin(2t) >$ and at $t = \pi$ this is $< 4, 8, 0 >$.

So, $B = \frac{1}{\sqrt{80}} < 4, 8, 0 >= \frac{1}{\sqrt{5}} < 1, 2, 0 >$.

(b) Calculate $\int_C (x + y + z)\, ds$ where C is the space curve described above as t varies from 0 to 2π.

Answer:

$$\int_0^{2\pi} (\sin(2t) - t + \cos(2t))\sqrt{5}\, dt = -2\sqrt{5}\pi^2.$$

(c) Calculate the integral $\int_C F \cdot dr$ where C is the space curve described above as t varies from 0 to 2π and $F =< 0, y, z >$.

Answer: We want

$$\int_C y\, dy + z\, dz = \int_0^{2\pi} t\, dt - 2\sin(2t)\cos(2t)\, dt.$$

This evaluates to $2\pi^2$.

3. 10 points

Grading Policy: $6 + 3 + 6 = 15$ points. Same general grading policy.

(a) Given the function

$$f(x, y, z) = 5\,xz - 3\,xy - 4\,yz$$

and the substitution:

$$x = uv, y = u^2 + v^2\ ,\ z = u - v,$$

calculate $\dfrac{\partial f}{\partial u}$. **You may** leave the answer in terms of x, y, z, u, v.

Answer: $(5\,z - 3\,y)\,v + 2\,(-3\,x - 4\,z)\,u + 5\,x - 4\,y$.

(b) Evaluate (and simplify) the above answers when $u = 1, v = -1$. Here, the final answer **must be** a number.

Answer: At this value, $x = -1, y = 2, z = 2$ so the answer evaluates to -27.

(c) Consider the function $F(x, y) = x^2 + txy + ty^2$, where t is a constant. The point $(0, 0)$ is clearly a critical point for the function $F(x, y)$, for all values of t.

Answer the following. **Hint:** You may ignore the values of t for which the second derivative test is inconclusive.

i. Determine all values of t (if any) for which, $(0, 0)$ is a saddle point. Explain your reasoning. **Hint:** Calculate the D at $(0, 0)$.
Answer: $D = 4t - t^2$. This is negative when t is negative or bigger than 4. Such points are saddle points.

ii. Determine all points (if any) for which $(0, 0)$ is a local **minimum.** Explain your reasoning.
Answer: $F_{xx}(0, 0) = 2 > 0$ and $D > 0$ when $0 < t < 4$. So these are the values where we have a local minimum.

iii. Determine all points (if any) for which $(0, 0)$ is a local **maximum.** Explain your reasoning. **Answer:** Since $F_{xx} > 0$ we will not have a local maximum at any value of t.

4. 10 points

Grading Policy: $4+2+2+2=10$ points. Same general grading policy.

Consider the function

$$f(x,y,z)=\frac{x}{y}+2\,\frac{y}{z}-2\,\frac{z}{x}.$$

Answer the following questions:

(a) Calculate $\nabla(f)$ and the Laplacian $\nabla^2(f)=\nabla\cdot\nabla(f)$.

Answer:

$$\nabla(f)=<\frac{1}{y}+2\,\frac{z}{x^2},-\frac{x}{y^2}+2\,\frac{1}{z},-2\,\frac{y}{z^2}-2\frac{1}{x}>$$

The Laplacian is

$$-4\,\frac{z}{x^3}+2\frac{x}{y^3}+4\,\frac{y}{z^3}$$

(b) Calculate $\nabla(f)(1,1,-1)$ and $\nabla\cdot\nabla(f)(1,1,-1)$.

Answer: $<-1,-3,-4>$ and 2.

(c) Find the directional derivative of f in the direction $<3,-1,-1>$ at the point $P(1,1,-1)$.

Answer: $\frac{4}{\sqrt{11}}=1.206$.

(d) Find the direction u in which the directional derivative of f at the point $P(1,1,-1)$ (i.e. $D_uf(1,1,-1)$) is minimized. What is the value of this minimum derivative?

Answer: The direction is $<1,3,4>$ and the value is $-\frac{26}{\sqrt{26}}=-\sqrt{26}$.

5. 10 points

Grading Policy: 4 points for sketch, 3 points to set up and 3 points for the final answer.

Consider the part of the paraboloid

$$z = 7 - 3x^2 - 3y^2 \textbf{ lying above the plane } z = 4.$$

Answer the following questions:

(a) **Carefully sketch** the surface and clearly mark the region of integration.
Answer: The area lies above the disc inside $4 = 7 - 3x^2 - 3y^2$ or $x^2 + y^2 = 1$ under the paraboloid.

(b) **First set up** the double integral for the surface area in terms of x, y. Describe the region of integration, but it is not necessary to identify the limits here.
Answer:

$$\int\int_D \sqrt{36x^2 + 36y^2 + 1}\, dA$$

where D is the area inside the disc $x^2 + y^2 = 1$.

(c) **Now evaluate** the integral **using an appropriate change of variables.** The limits must be clearly stated here.
Answer: Using polar coordinates, we get

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{36r^2 + 1}\, r\, dr\, d\theta = 2\left(\frac{37}{108}\sqrt{37} - \frac{1}{108}\right)\pi = 13.03541119.$$

6. 10 points

Grading Policy: 3 points for sketch and 3 points for the integral. Give 3 points for calculation and 1 point for the last part.

(a) **Carefully sketch** the prism lying above the triangle with corners $A(0,0,0), B(2,0,0), C(0,1,0)$ and bounded by the planes $z = 1$ and $z = 5$. Call the resulting solid R.

(b) Suppose that the solid R has density given by $\rho(x, y, z) = xyz$.
Suppose we calculate the mass of the solid R using the integral

$$\int_{z=a}^{b} \int_{y=c}^{d} \int_{x=e}^{f} (xyz) dx\, dy\, dz.$$

Determine the correct limits for this integral and write out the correct expression of the integral.
Answer: $\int_{z=1}^{5} \int_{y=0}^{1} \int_{x=0}^{2-2y} (xyz) dx\, dy\, dz.$

(c) Determine **the mass** of the solid region R. Be sure to simplify the final answer.
Answer: The answer is 2.

(d) Set up the formula for $\overline{z}$, the z-coordinate for the center of mass. Your answer should involve an explicit integral and may use the mass found above. **It is not necessary** to evaluate the integral.
Answer: $\frac{1}{2} \int_{z=1}^{5} \int_{y=0}^{1} \int_{x=0}^{2-2y} (xyz^2) dx\, dy\, dz.$

7. 10 points

Grading Policy: $4+3+3=10$ points. Same general grading policy.

(a) Given the vector field

$$F =< x^2 - xy, y^2 - 2\,xy, 2\,xyz >$$

calculate curl F and div F.

Answer: curl $F =< 2\,xz, -2\,yz, -2\,y + x >$, div $F = y + 2\,xy$.

(b) Is F a conservative field? Why?

If it is a conservative field, find a function f such that $\nabla f = F$.

Answer: No, because curl $F \neq 0$.

(c) Given the function $g(x, y, z) = \mathrm{e}^z\,(x + 2\,y)$ compute its gradient $\nabla(g)$ and its Laplacian $\nabla \cdot \nabla(g)$.

Answer: $\nabla(g) =< \mathrm{e}^z, 2\,\mathrm{e}^z, \mathrm{e}^z\,(x+2\,y) >$. $\nabla \cdot \nabla(g) = \mathrm{e}^z\,(x + 2\,y)$.

8. 10 points

Grading Policy: 2 points for explanation and 4 points for the sketch, 4 points for the final answer.

Use Green's Theorem to evaluate the line integral

$$\int_C (\exp(x)\sin(2x) + 2y)dx + (2x + \cos(3y) + y^2)dy$$

where C the positively oriented boundary of the region bounded by the two parabolas:

$$y = 9 - x^2 \text{ and } y = -9 + x^2.$$

Carry out the following instructions. Each part will be graded.

- Carefully sketch the region.
- Be sure to show the boundary curve and mark the orientation.
- Explain why the Green's theorem is applicable.

Answer: If we write the integral as $\int_C P\,dx + Q\,dy$ then it equals $\int_D (Q_x - P_y)dA$. But $Q_x - P_y = 0$. So the evaluation is 0. Also, the hypothesis of green's theorem is valid since all the derivatives are continuous, so the answer for the line integral is 0.

9. 10 points

Grading Policy: 3 points for explanation 3 points to find the F, 4 points for the final answer.

Consider the integral

$$\int_C (3x^2y^3 + 4x^2)dx + (3x^3y^2 + 5y)dy$$

where C is any simple continuous curve in the plane.

Answer the following:

(a) In your own words, briefly explain what it means to say the the above line integral is independent of path. Be sure to write complete correct sentences.

(b) Prove that the above integral is independent of path.
Hint: Find a function $F(x, y)$ such that

$$(3x^2y^3 + 4x^2)dx + (3x^3y^2 + 5y)dy = dF = F_x dx + F_y dy.$$

(c) Use this to find the value of the integral when C is some path from $P(0, 0)$ to $Q(3, 4)$.
Answer: The function F is $x^3y^3 + 4x^3/3 + 5y^2/2$. Hence the integral is $F(3, 4) - F(0, 0) = 1804$.