1 Vector Functions or space curves.

Basic formulas. Starting notation: $P(t)$ (Position function) or $\mathbf{r(t)}$ (coordinate function).

$v(t) = \mathbf{r}'(\mathbf{t})$ velocity or tangent direction $\mathbf{a(t)} = \mathbf{r}''(\mathbf{t})$ acceleration or second derivative.

1. $\sigma(t) = |v(t)|, T(t) = \dfrac{v(t)}{\sigma(t)}$.
2. $s(t) = \displaystyle\int_0^t \sigma(\tau)\, d\tau$. Arclength/distance traveled.
3. $a(t) = \sigma'(t)T(t) + \sigma^2(t)\kappa(t)N(t)$.
4. $H(t) = v(t) \times a(t) = \kappa(t)\sigma^3(t)B(t)$, and so $\kappa(t) = \left|\dfrac{H(t)}{\sigma^3(t)}\right|$.
5. $N(t) = B(t) \times T(t) = \dfrac{H(t) \times v(t)}{|H(t)||v(t)|} = \dfrac{H(t) \times v(t)}{\kappa(t)\sigma^4(t)}$.
6. Tangential component of acceleration $= \sigma'(t) = \frac{v(t)\cdot a(t)}{|v(t)|}$.

   Normal component of acceleration$= \frac{|H(t)|}{|v(t)|}$.

Plane motion. If $B(t)$ is a constant vector, then the curve lies in a plane with normal $B(t)$. The equation of the plane is then computed as $B(t) \cdot (X - P(t_0)) = 0$. A simpler formula is $H(t_0) \cdot (X - P(t_0)) = 0$. Either of these formulas work at some $t = t_0$ for which $H(t_0) \neq 0$.

Tangents. The tangent line to a space curve $X = \mathbf{r(t)}$ at $t = p$ is a line with direction vector $v(p) = \mathbf{r}'(\mathbf{p})$ and passing through the point $\mathbf{r(p)}$.

A normal line and a binormal line at the same point is found by using the same point and direction vectors $N(p)$ and $B(p)$. In practice, it is more convenient to use the vectors $H(p) \times v(p)$ and $H(p)$.

The osculating plane at the point is a plane passing through the point and with normal $B(p)$.

2 Functions of Several Variables.

All formulas are samples and need to be adjusted for number of variables.

basic Formulas.

1. Universal derivative $D(f(x, y, z)) = f_x D(x) + f_y D(y) + f_z D(z)$.
2. Basic rules $D(a) = 0$ if a is a constant, $D(f + g) = D(f) + D(g)$, $D(fg) = fD(g) + gD(f)$.
3. Basic rules for vectors of functions. $D(F \cdot G) = D(F) \cdot G + F \cdot D(G)$. $D(F \times G) = D(F) \times G + F \times D(G)$.
4. Gradient $\nabla f(x, y, z) =< f_x, f_y, f_z >$.
5. Directional derivative $D_v(f) = \nabla(f) \cdot \frac{v}{|v|}$.

6. From the universal derivative: If we have functions A, B, C such that $D(f) = AD(x) + BD(y) + CD(z)$, then $\nabla(f) =< A, B, C >$.

Linear Approximation of a function. Given: $f(X)$ where X stands for many variables, e.g. $X = (x, y, z)$ and a point A, e.g. $A = (a, b, c)$. Then $f(A + \Delta(X)) \approx f(A) + \nabla(f) \cdot \Delta(X)$. Writing $X = A + \Delta(X)$, we have

$$\text{Linear approximation } L(X) = f(A) + \nabla(f)(A) \cdot (X - A).$$

The approximation is valid provided $\nabla(f)$ is continuous at A and $\Delta(X) = X - A$ is small enough.

Linear Approximation of an implicit function. If we have a relation $F(x, y, z) = 0$ then we **estimate**

$$\Delta(F)(a, b, c) = F_x(a, b, c)\Delta(x) + F_y(a, b, c)\Delta(y) + F_z(a, b, c)\Delta(z) = 0$$

provided $\nabla(F)$ is continuous at (a, b, c) and $\Delta(x), \Delta(y), \Delta(z)$ are small enough. Then given any two of $\Delta(x), \Delta(y), \Delta(z)$, we can estimate the third, provided the partial derivative with respect to it is not zero.

3 Tangents.

Given a surface $f(x, y, z) = \lambda$ where λ is a constant, and a point (a, b, c) the tangent plane is the plane through (a, b, c) with normal $\nabla(f)(a, b, c)$.

For the graph of a function $z = F(x, y)$ we make the equation $z - f(x, y) = 0$ and at $x = a, y = b$, we get

$$< -f_x(a, b), -f_y(a, b), 1 > \cdot < x - a, y - b, z - f(a, b) >= 0.$$

A surface also has a normal line, namely the line through the point and normal to the tangent plane.

4 Critical Points and Max/min.

For a function $z = f(x, y)$ a point $(x, y) = (a, b)$ is a critical point if $\nabla(f)(a, b) =< 0, 0 >$.

At a critical point, all directional derivatives are zero.

Technically points where $\nabla(f)$ is undefined is also included here, but the claim about derivatives is not meaningful, since they may fail to exist.

A point is a local max or min **only if** it is critical.

But a critical point may be a local max or a local min or neither.

We have a second derivative test:

- Assume (a, b) is a critical point.
- Assume that f_{xx}, f_{xy} and f_{yy} are continuous near (a, b) and let their values be p, q, r respectively and at least one of them is non zero.
- Let $D = pr - q^2$.
- The point is a local min if p and D are both positive. The point is a local max if $D > 0$ and $p < 0$. The point is a saddle point if $D < 0$ (and neither local max, nor local min).
- If $D = 0$, then the test is inconclusive. We need to study higher order tests.