We generally assume our functions to be well behaved, i.e. having sufficiently many continuous partial derivatives in the region of interest. The curves and surfaces used are also assumed to be sufficiently piecewise smooth.

1 Plane Line Integrals.

A plane curve C is assumed parametrized as $r(t) =< x(t), y(t) >$ where t varies in $[a, b]$.

1. For a function $f(x, y)$, we define integral of f along C by:
$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt.$$
2. For a vector field $F(x, y) =< P(x, y), Q(x, y) >$ we define integral of F along C by:
$$\int_C F\cdot T\,ds = \int_C F(x(t),y(t))\cdot T(t)\sqrt{x'(t)^2+y'(t)^2}\,dt = \int_a^b F(x(t),y(t))\cdot < x'(t),y'(t) > \,dt.$$
Thus, it is also seen to be $\int_C F\cdot dr = \int_C F\cdot < dx, dy >$.
3. A vector field F is conservative if $F = \nabla(f)$ for some function f. A conservative field F has the property that
$$\int_C F\cdot dr = f(B) - f(A)$$
for any curves C from A to B.

   In particular, for a closed curve, the integral is zero; provided the functions involved are continuous in an open domain containing the curve.
4. A necessary condition for a field $F =< P, Q >$ to be conservative is that $P_y = Q_x$. This is sufficient provided **the functions are well behaved (have continuous partial derivatives) in a simply connected domain** and we restrict our curves to that domain.

2 Space Line Integrals.

1. Integral of a function $f = f(x, y, z)$ on the curve C is defined by:
$$\int_C f\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt.$$
2. Integral of a vector field F along a curve C is defined by:
$$\int_C F\cdot T\,ds = \int_a^b F(x(t),y(t),z(t))\cdot < x'(t),y'(t),z'(t) > \,dt.$$
As before, this is seen to be $\int_C F\cdot dr = \int_C F\cdot < dx, dy, dz >$.

3. A conservative field is defined similarly, except the function f is in three variables. the result about the integral of a conservative field being dependent only on the end points is still valid.

4. A necessary condition for a field to be conservative is that curl $F = 0$ where:

$$\text{curl } F = \nabla \times F = < R_y - Q_z, P_z - R_x, Q_x - P_y > \text{ where } F = < P, Q, R >.$$

   The sufficiency again requires the simple connectedness (no holes).

5. Another useful function is "divergence" $\nabla \cdot F =$ div $F = P_x + Q_y + R_z$ where F is as above. It satisfies the equation: div (curl F) $= 0$ provided the functions have continuous second order derivatives (i.e. Clairaut's theorem is applicable).

6. The Laplacian is defined as $\nabla \cdot \nabla(f) = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$.

3 Surface Integrals.

1. If S is a parametrized surface $r(u, v) =< x(u, v), y(u, v), z(u, v) >$ where $(u, v) \in D$, then integral of a function f on S is:

$$\int\int_S f(x, y, z)\, dS = \int\int_D f(r(u, v))|r_u \times r_v|\, dA.$$

2. In particular, the surface area is given by:

$$\int\int_S dS = \int\int_D |r_u \times r_v|\, dA.$$

   As usual, we can integrate density function for the mass, or other suitable functions are inserted for moments etc.

3. For a surface defined by $g(x, y, z) = 0$, we get:

$$\int\int_S f(x, y, z)\, dS = \int\int_D f|\nabla(g)|\, \frac{dx\, dy}{|g_z|}.$$

   provided $g_z \neq 0$.

4. We define the **Fundamental differential** on S to be:

$$\omega_g = \frac{dx\, dy}{g_z} = \frac{dy\, dz}{g_x} = \frac{dz\, dx}{g_y}.$$

   This can be used to change the formula to any convenient projection. This gives a better formula:

$$\int\int_S f(x, y, z)\, dS = \int\int_D f|\nabla(g)|\, |\omega_g|.$$

5. The integral of a vector field F on the surface $g(x, y, z) = 0$ is given by

$$\int\int_S F \cdot \overrightarrow{dS} = \int\int_S F \cdot \overrightarrow{n}|\nabla(g)|\, |\omega_g| = \int\int_D F \cdot \nabla(g)\, |\omega_g|.$$

   For a parametrized surface, this becomes;

$$\int\int_D f(r(u, v)) \cdot r_u \times r_v\, dA.$$

4 Theorems.

1. **Green's Theorem.** If C is a positively oriented piecewise smooth closed plane curve enclosing a domain D and P, Q have continuous partial derivatives on a domain containing C, then

$$\int_C P\,dx + Q\,dy = \int\int_D (Q_x - P_y)\,dA.$$

2. **Stokes' Theorem.** If C is a positively oriented piecewise smooth closed space curve enclosing a positively oriented piecewise smooth surface S and F is a vector field having continuous partial derivatives in a domain containing S, then

$$\int_C F \cdot dr = \int\int_S \text{curl } F \cdot \overrightarrow{dS}.$$

Here, if the surface is defined by $g(x, y, z) = 0$, then $\overrightarrow{dS} = \nabla(g)|\omega_g|$ where

$$|\omega_g| = \left|\frac{dx\,dy}{g_z}\right| = \left|\frac{dy\,dz}{g_x}\right| = \left|\frac{dz\,dx}{g_y}\right|$$

In parametric form, $\overrightarrow{dS}$ comes out as $r_u \times r_v\,dA$.

Be sure to review the formulas from the first three exams.