Three Dimensional Space.

1. **Coordinates.**

   Just like the usual plane, we can imagine a three dimensional space with three pairwise perpendicular axes called the x, y and z axis respectively.

   The sketch of three dimensional objects is complicated to draw and read, since of necessity the objects are projected into the plane where we have to plot them.

   Points in three space are still given by their three coordinates, e.g. $P(1, 2, 3)$.

2. **Planes and Lines.**

   A single linear equation now takes on the form $ax+by+cz = d$, for example $x+2y+3z = 6$. The resulting locus is a "plane". To sketch it, we typically plot its intersections with the three axes $(6, 0, 0), (0, 3, 0), (0, 0, 2)$ and join them into a triangle. The plane is imagined to stretch from it in all directions.

   What does a line look like? It is no longer given by one equation. For example, the x-axis is described as $y = 0, z = 0$ which are equations of two planes.

   A better description is $\{(t, 0, 0)|t \in \Re\}$ and this is called the parametric form. We shall find such a form to be a lot more manageable.

3. **Distance Formula.** The distance between two points $P(a_1, b_1, c_1), Q(a_2, b_2, c_2)$ is given by

   $$d(P, Q) = |PQ| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}.$$

   A sphere (analog of a circle) with center $P(a, b, c)$ and radius r is given by

   $$\{Q||PQ| = r\} \text{ or } \{Q(x, y, z)|(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2\}.$$

4. **Inequalities.** An inequality like $ax + by + cz < d$ describes all points on one side of the plane $ax + by + cz = d$. If we change $<$ to $>$, then we get the other side.

5. **Cylinders.** A general surface is given by an equation of the form $f(x, y, z) = c$ where c is some constant. If the expression $f(x, y, z)$ is missing one of x, y, z, then the resulting surface is seen to be a cylinder (with some possibly complicated base) parallel to the axis corresponding to the missing variable.

   It is not necessary for a variable to be missing, but this can be arranged after a change of variables!

Vectors

1. **Basic Vector Calculations.**

   Typically, vectors in Euclidean spaces $\Re^n$ appear as n-tuples of real numbers. Thus a plane vector looks like $< 3, 4 >$ and three dimensional vector looks like $< -2, 3, 7 >$. etc.

   To add two vectors, they must be of the same length and then we add them componentwise.

   $$< 1, -2, 3 > + < 4, 2, 5 >=< 5, 0, 8 > .$$

We can also scale a vector by multiplying it by a number c and so such numbers are called **scalars**. The operation is called **scalar multiplication.**

Thus $4 < 1,2,3 > = < 4,8,12 >$ ..

These two operations can be combined to make new vectors.

$$2 < 1,2,-2 > -3 < 1,0,2 > = < -1,4,-10 > .$$

These concepts can be generalized to abstract vector spaces, studied in Linear Algebra. We will discuss them but not clutter these notes with the abstract concepts.

2. Geometry Associated With vectors.

   A vector $< 1,2 >$ may be interpreted as a "displacement" of 1 unit in the x-direction followed by 2 units in the y-direction. Thus, if we start at the point $A(-2,5)$ and apply the displacement, then we reach $B(-1,7)$ since we see that $-2+1=-1$ and $5+2=7$. So, we could express our vector as a displacement from A to B, described in symbols as $\overrightarrow{AB} = < 1,2 >$.

   By a slight abuse of notation, we shall write

   $$\overrightarrow{AB} = B - A = < 1,2 >$$

   .

   It is evident that many different point pairs will give the same vector and geometrically these different point pairs can be imagined as obtained by moving an arrow stretched from A to B anywhere in the plane without changing its direction.

   The vector addition can now be interpreted by saying

   $$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

   Check this by noticing a simple calculation trick:

   $$\overrightarrow{AB} = B - A,\ \overrightarrow{BC} = C - B,\ \overrightarrow{CA} = A - C \text{ so } (B-A)+(C-B)=(C-A).$$

   If we multiply a vector v by -1, then it is clear that we simply reverse the arrow

   $$(-1)\overrightarrow{AB} = -(B-A) = A - B = \overrightarrow{BA}.$$

   More generally multiplication by c multiplies the length by $|c|$ and reverses the direction exactly when $c < 0$.

3. **General Vectors.** More generally, vectors model quantities which have a built in direction and magnitude. A formal mathematical definition stipulates that a vector space is a non empty set with a concept of addition and scalar multiplication satisfying certain natural axioms. [1]

   Another typical interpretation, besides displacement, is forces acting on object.

[1] We list these axioms without much explanation. You should consult a book on linear algebra for more information.

- The addition is commutative, has a "zero" and negatives.
- The scalar multiplication is unitary. $1v = v$.
- There are natural identities:

$$(u+v)+w = u+(v+w), v+0 = v, c(d(v)) = (cd)v, c(u+v) = cu+cv, (c+d)u = cu+du.$$

4. **Standard bases.** It is customary to name some special vectors:

$$\mathbf{i} =< \mathbf{1,0,0} >, \mathbf{j} =< \mathbf{0,1,0} >, \mathbf{k} =< \mathbf{0,0,1} > .$$

   These are a basis for the three space.

   Then $< 1, -2, 5 >= \mathbf{i} - \mathbf{2j} + \mathbf{5k}$.

   This helps in writing and simplifying some expressions. Occasionally, we will abuse the notation and use $\mathbf{i}, \mathbf{j}$ to denote a basis in the plane, where the third coordinate is dropped.

5. **Dot Product.** We mentioned that vectors have magnitudes. What is magnitude (length) of a vector? If we think of the vector as a displacement, then it should the the distance between the starting and ending points.

   Thus we may write

$$|\overrightarrow{AB}| = d(A, B).$$

   Our distance formula then yields

$$| < a, b, c > | = \sqrt{a^2 + b^2 + c^2}.$$

   We now define a "dot product " of two vectors by the formula:

$$< a_1, b_1, c_1 > \cdot < a_2, b_2, c_2 >= a_1a_2 + b_1b_2 + c_1c_2.$$

   With this notation, we see that $v \cdot v = |v|^2$.

   But what possibly is $v \cdot w$ where w may be different from v?

   it can be shown that

$$v \cdot w = |v||w| \cos(\theta) \text{ where } \theta \text{ is the angle between } v, w.$$

   [2]

   In abstract vector spaces, the above formula is interpreted as giving a definition of the angle between two vectors, whereas in ordinary Euclidean spaces, it can be deduced from Euclidean Geometry.

6. Properties of the Dot Product.

   Various properties can be deduced using the formula applied to general vectors. In abstract spaces, they are a consequence of the axioms.

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[2]In abstract vector spaces, this is turned around completely. We define an abstract dot product, called an inner product which satisfies the conditions:

$$u \cdot v = v \cdot u, u \cdot (v + w) = u \cdot v + u \cdot w, cu \cdot v = (cu) \cdot v = u \cdot (cv)$$

and moreover

$$u \cdot u \geq 0 \text{ where equality happens exactly when } u = 0.$$

Then $u \cdot u$ defines the square of an abstract length and the above formula **defines** and angle between the vectors. The fact that $\frac{u \cdot v}{|u||v|}$ is the cosine of an angle is guaranteed by the famous Cauchy-Schwartz inequality!

(a) Let v be a non zero vector. Then either of the two vectors $\pm\frac{v}{|v|}$ is a unit vector. These are said to be unit vectors parallel to v (along v) if we use the plus sign and antiparallel (opposite v) in the case of the minus sign.

(b) The angle between two vectors v, w is computed as $\arccos(\frac{v\cdot w}{|v||w|})$. Note that the vectors need to be non zero for this to make sense.

(c) $v \cdot w = 0$ iff the angle between them is $\pi/2$ radians, or one of them is a zero vector. This becomes a very effective test of perpendicularity of two vectors.

(d) $|v \cdot w| = |v||w|$ iff the vectors are either parallel or antiparallel, or one of them is zero.

(e) $(av + bw) \cdot h = (a)v \cdot h + (b) \cdot h$.

(f) Given non zero vectors v, w, we can ask for a vector of the form $v - cw$ which is perpendicular to w. this is easily found by solving

$$(v - cw) \cdot w = v \cdot w - cw \cdot w = 0$$

and gives $c = \frac{v\cdot w}{w\cdot w}$. Note that this answer makes sense since w is a non zero vector.
The vector cw is parallel (or antiparallel) to w and is seen to give the foot of the perpendicular from tip of v onto the line containing w. This is called the **projection** of v on w.
This is a very useful operation with many applications. Note that it is valid in any dimensions!

(g) If we note that $u = \frac{w}{|w|}$ is a unit vector along w, then we can write the projection as:

$$proj_w v = \left(\frac{v \cdot w}{w \cdot w}\right) w = \left(\frac{v \cdot w}{|w|^2}\right) w = \left(\frac{v \cdot w}{|w|}\right) \frac{w}{|w|} = \left(\frac{v \cdot w}{|w|}\right) u.$$

We define the quantity $\left(\frac{v\cdot w}{|w|}\right)$ to be the component of v along w and denote it as $comp_w\, v$. This is also denoted as "the scalar projection" of v on w.

(h) Note that for a vector $v =< x, y, z >$, we can interpret

$$x = comp_{\mathbf{i}}\, v, y = comp_{\mathbf{j}}\, v, z = comp_{\mathbf{k}}\, v.$$

(i) It is obvious that for a vector $v =< x, y, z >$ we have

$$x = v \cdot \mathbf{i} = |\mathbf{v}| \cos(\alpha), \mathbf{y} = \mathbf{v} \cdot \mathbf{j} = |\mathbf{v}| \cos(\beta), \mathbf{z} = \mathbf{v} \cdot \mathbf{k} = |\mathbf{v}| \cos(\gamma)$$

where α, β, γ are respectively the angles made by v with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. These angles are called the direction angles and their cosines as the direction cosines.
This gives a different representation of the vector as

$$v = |v| < \cos(\alpha), \cos(\beta), \cos(\gamma) > .$$

This is a generalization of polar coordinates to three space and it can be extended to any dimensions.
Note the identity deduced from $v \cdot v = |v|^2$, namely

$$1 = \cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma).$$

This characterizes all possible triples of direction angles. [3]

[3] All this can be generalized to abstract higher dimensional vector spaces with a proper notion of inner (dot) products.

7. **The Cross Product.** In **three dimensions only,** there is another concept of a product of vectors. Given two vectors

$$v = a_1\mathbf{i} + \mathbf{b_1 j} + \mathbf{c_1 k} \text{ and } \mathbf{w} = \mathbf{a_2 i} + \mathbf{b_2 j} + \mathbf{c_2 k},$$

their cross product is defined by

$$v \times w = (b_1c_2 - b_2c_1)\mathbf{i} + (\mathbf{c_1a_2} - \mathbf{c_2a_1})\mathbf{j} + (\mathbf{a_1b_2} - \mathbf{a_2b_1})\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

Note that the formula has a pleasing symmetry and the determinant form is useful in remembering it.

8. **A simple Observation.**

Given any vector $u = a_0\mathbf{i} + \mathbf{b_0 j} + \mathbf{c_0 k}$ it is easy to verify that

$$u \cdot (v \times w) = (b_1c_2 - b_2c_1)a_0 + (c_1a_2 - c_2a_1)b_0 + (a_1b_2 - a_2b_1)c_0 = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

The last determinant makes it clear that if $u = pv + qw$ for any scalars p, q, then $u \cdot (v \times w) = 0$.

So, our cross product $v \times w$ is always perpendicular to both v, w. [4]

It is also evident that if v, w are linearly dependent, then $v \times w = 0$ since all determinants in the expression are zero.

Thus, when v, w are linearly independent (which means that their combinations fill out a plane), then $v \times w$ is naturally a vector perpendicular to both v, w and in three space all such vectors are multiples of a single non zero vector. Our cross product produces a natural vector with this property.

It does a lot more.

We have:

- $$|v \times w| = |v||w||\sin(\theta)| \text{ where } \theta \text{ is measured counterclockwise from } v \text{ towards } w.$$

  This description needs a careful thought, since it depends on the direction in which we observe the plane. The cross product, indeed, fixes a direction which can be defined as naturally the "up" direction for this purpose.

  This is often described as the "right hand rule" which says that if you hold your right hand so that the index finger points along v and the middle finger along w, then the thumb gives the direction of $v \times w$. Of course, one has to draw the axes correctly so that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

  The formula $|v||w|\sin(\theta)$ gives the signed area of the parallelogram formed from vectors v, w, where we get a positive sign if the direction from v to w is counterclockwise as viewed from up (i.e. in the direction of v) and negative if it is clockwise as viewed from up.

[4] Here, for technical convenience, we consider a zero vector as perpendicular to any vector.

- $$v \times w = -w \times v, v \times (p+q) = v \times p + v \times q, c(v \times w) = (cv) \times w = v \times (cw).$$

- $$u \cdot v \times w = u \times v \cdot w = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

  in the above notation. Moreover this "triple product" is equal to $|u||v||w|\sin(\theta)\cos(\alpha)$, where θ is the angle from v towards w as explained above and α is the angle made by u with the plane spanned by v and w.

  It gives the volume of the parallelepiped with u, v, w as the corner vectors and it is a signed volume, in the sense that it is positive exactly when the vectors are independent and u points up from the v, w plane.

  This last statement is often paraphrased as the "right hand rule" saying that if you hold your right hand with the index and the middle finger in the direction of v, w respectively, then the thumb can be made to point towards u without breaking your hand!

- Finally, we consider $h = u \times (v \times w)$. Unlike the other triple product, here the answer is more complicated. First we note that the h is a vector perpendicular to u as well as $v \times w$. Assuming independent vectors, we see that anything perpendicular to $v \times w$ must be a combination of v, w, so the answer has the form

  $$h = av + bw \text{ where } a(u \cdot v) + b(u \cdot w) = 0.$$

  One obvious solution is

  $$h = (u \cdot w)v - (u \cdot v)w.$$

  This must be the correct answer except for a scalar factor and indeed it can be verified by explicit calculation.

9. **Lines in Space.** A line is determined by two points P_0, P_1. Suppose that we take vectors $\overrightarrow{OP} = \mathbf{r_0}, \overrightarrow{OQ} = \mathbf{r_1}$ starting at the origin O with tips at P_0, P_1, respectively. Then it is not hard to see that the vector pointing to a random point R on the line P_0P_1 is given by

$$\overrightarrow{OR} = \mathbf{r} = \mathbf{r_0} + \mathbf{t}\left(\mathbf{r_1} - \mathbf{r_0}\right).$$

Moreover, this parameter t can be taken as a coordinate on the line with P_0 having coordinate 0 and P_1 having coordinate 1.

If we are in three space and use coordinates x, y, z, then this can be described in a natural notation by:

$$x = a_0 + t(a_1 - a_0), y = b_0 + t(b_1 - b_0), z = c_0 + t(c_1 - c_0).$$

If we are bold, then we can rewrite this in a symmetric form:

$$t = \frac{x - a_0}{a_1 - a_0} = \frac{y - b_0}{b_1 - b_0} = \frac{z - c_0}{c_1 - c_0}.$$

The boldness involves in **allowing a zero denominator with the understanding that then so is the numerator.**

The same equation used in the plane gives a different way of finding an equation of a line through $P_0(a_0, b_0), P_1(a_1, b_1)$ as:

$$t = \frac{x - a_0}{a_1 - a_0} = \frac{y - b_0}{b_1 - b_0} \text{ or } (b_1 - b_0)(x - a_0) = (a_1 - a_0)(y - b_0).$$

To be continued ...