1 Equations.

We now discuss equations defining geometric objects in three space. Our notation and calculations would naturally be valid in n-dimensional spaces, where feasible.

We will generally denote by $\mathbf{r}$ the position vector to a general point of our geometric object. Thus, $\mathbf{r} =< \mathbf{x}, \mathbf{y}, \mathbf{z} >$ in three space and $\mathbf{r} =< \mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n} >$ in general.

1. **Point.** Then a single point is given by $\mathbf{r} = \mathbf{r_0}$ where $r_0 =< a, b, c >$. Explicitly, a point $P(1, 2, -4)$ is given by $\mathbf{r} =< \mathbf{1}, \mathbf{2}, -\mathbf{4} >$.

2. **Line.**

   A line is given by $\mathbf{r} = \mathbf{r_0} + \mathbf{tv}$ where t is a parameter, v is a vector lying in the direction of the desired line and $\mathbf{r_0}$ is a specific point on it. [1]

   Thus a line has one free parameter t and it appears linearly.

3. **Curve.** More generally, a curve in space is given by $\mathbf{r} = \mathbf{r_0} + \mathbf{f(t)}$ where, $f(t)$ is a vector of functions of t - $< f_1(t), f_2(t), \cdots, f_n(t) >$.

   We shall study these in detail later.

4. **Plane.**

   A plane can be defined as $\mathbf{r} = \mathbf{r_0} + \mathbf{sv} + \mathbf{tw}$ where v, w are two vectors which are independent (neither is a multiple of the other) and s, t are parameters. A little thought will show that in two dimensions, every point is included in the resulting set.

   In three dimensions, we have equations

   $$x = a + sv_1 + tw_1, y = b + sv_2 + tw_2, z = c + sv_3 + tw_3.$$

   A little calculation shows that $\mathbf{r} - \mathbf{r_0}$ is a combination of v, w and thus in three space, we have

   $$(\mathbf{r} - \mathbf{r_0}) \cdot v \times w = 0.$$

   This gives a single equation involving x, y, z and is the usual form of the equation of a plane.

   The vector $\mathbf{r} - \mathbf{r_0}$ is seen to be a vector which "lies" in the plane since it joins two points in the plane. Thus, the equation shows that $v \times w$ is a vector perpendicular to every vector in the plane. It is then called a normal to the plane.

   Often, an equation of a plane is written as $\mathbf{n} \cdot < \mathbf{x} - \mathbf{a}, \mathbf{y} - \mathbf{b}, \mathbf{z} - \mathbf{c} >= \mathbf{0}$, where $< a, b, c >$ is any point in the plane and $\mathbf{n}$ is its normal.

5. **Surface.** More generally, an equation $f(x, y, z) = 0$ describes a "surface" in the three space. A more convenient form of this is $z - g(x, y) = 0$, but this may not be available or convenient. We concentrate on some special surfaces called the quadric surfaces (degree two equations) next. Their study is analogous t the study of conic sections.

[1] Compare this with the usual "point slope " form: $y - b = m(x - a)$. It can be rewritten as $< x, y >=< a, b > +t < 1, m >$ which gives $y = b + tm = b + m(x - a)$! Note that our form is more powerful since it makes no exception for a vertical line.

2 Quadric Surfaces.

1. **Convenient restrictions.** We study a surface $f(x, y, z) = 0$ where f is a polynomial and all terms have degree at most 2. Ignoring the special case of a plane, we will also assume that all mixed terms, xy, yz, xz are missing. This can be arranged by a change of variables but we don't worry about these technicalities. Also, if an x^2 term is present, then we can remove the x term by a translation (completing the square process). [2]

   This leaves us with the following main types of equations where we have moved the constant term on the right.

   Cylinder If a variable is missing, then we get a cylinder whose axis is parallel to the axis of the missing variable. For example: $(x-1)^2 + (y-1)^2 = 25$ is a cylinder parallel to the z-axis with base as a circle with center $(1, 1)$ and radius 5 in the xy-plane.

   Ellipsoid $px^2 + qy^2 + rz^2 = s$ where p, q, r are all positive. Naturally, we must have $s > 0$ also, for otherwise we get an empty or a one point surface. It is customary to rearrange such a surface by dividing by s and then writing all the positive quantities as squares. Thus we write: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

   Hyperboloid of one sheet If we have an equation like an ellipsoid but one of the terms is made negative then we get this surface. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. This is visualized by rewriting it as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$ and note that as z increases, we get horizontal ellipses which get bigger. All ellipses have centers along the z-axis and their axes expand proportionally.

   In this case, the z-axis is called the **the principal axis of the surface.** More generally, if the surface has the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 + \frac{(z-l)^2}{c^2}$, then the line $x = h, y = k, z = t$ is the principal axis.

   These are called hyperboloid because vertical sections by $x = p$ or $y = p$ are hyperbolas. It is called one sheeted because the whole surface is connected.

   Hyperboloid of two sheets If two terms of the ellipsoid are made negative, then the equation looks like $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. This is best visualized by setting $x = p$. Note that $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2} - 1$. So, for $|x| < 1$ we have no points, but all large values of $|x|$ produce ellipses. Sections by $y = p$ or $z = p$ produce hyperbolas and the surface has two components.

   The centers of the hyperbolas form the principal axis. Thus if the surface has the form $\frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = \frac{(x-h)^2}{a^2} - 1$, then the principal axis is the line $x = t, y = k, z = l$.

   Cone The above surfaces missed the origin. If we assume that the surface passes through the origin (after the preparation described above), then we get an equation $\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm + \frac{z^2}{c^2} = 0$. This surface reduces to just a point (origin) if all three signs are equal. Otherwise, one is minus and the other two are plus. For example, we may have: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$. Then a fixed

[2] For example, if we have $ax^2 + bx$ with $a \neq 0$, then we rewrite it as $a(x - \frac{b}{2a})^2 - \frac{b^2}{4a}$ and then we take $x - \frac{b}{2a}$ as a new x, so that it appears as $ax^2 - \frac{b^2}{4a}$.

nonzero value of z produces an ellipse and the surface is traced by lines joining its point to the origin.

Elliptic Paraboloid If one of the square terms, say z^2 is missing, then the equation has the form $\dfrac{z}{c} = \pm\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2}$. If both signs are the same, then they may be assumed to be both plus and all positive values of $\frac{z}{c}$ give ellipses while negative values of $\frac{z}{c}$ yield no points. Moreover, $x = p$ or $y = p$ produce parabolas, hence the name.

Hyperbolic Paraboloid This is similar to above but with mixed signs. So we may assume $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$. then fixed values of $\frac{z}{c}$ produce hyperbolas, while $x = p$ or $y = p$ produce parabolas. Hence the name. This surface has a very intriguing behavior at the origin.