1 Vector Functions.

1.1 Introduction.

1. We consider a vector in n-dimensions whose entries are functions. Typically, $n = 2$ or 3. The case of $n = 1$ is well studied in early calculus courses. The functions could be functions of more than one variables, but here we restrict attention to one variable.

   Thus, we study $\mathbf{r(t)} =< \mathbf{f_1(t), f_2(t), \cdots, f_n(t)} >$.

2. Such a vector can be interpreted as giving the position vector of a point in n-space corresponding to t. Thus, as t moves in a real line, the points trace out a curve in n-space. This gives a "space curve" parameterized by t.

3. As the movement of a particle, it makes sense to have a local coordinate system at each point which gives a good description of the movement at an instant.

   - For all dimensions, the direction of the movement is formally defined as the **tangent direction** which serves as the local "x" axis.
   - If you imagine facing this x-axis, then there may be a sense of turning to the left or right. It makes sense to think of that direction as the y-axis. For us, it is the **principal normal direction.**

     In the plane, these two directions give a local coordinate system.
   - In three space, there is a sense of "up" or z-axis. This is traditionally chosen by taking the cross product of the previous two direction, so it looks like the natural right handed frame. This is called the **binormal direction**.
   - In higher dimensions, we need more directions, but we shall not discuss these here. Also, there are degenerate situations which need to be recognized and handled.
   - We shall derive convenient formulas to calculate all the details from the derivatives of the position function $\mathbf{r(t)}$. We have a slightly different method of handling the formulas since we always treat the derivatives of $\mathbf{r(t)}$ as the main information, whereas the book starts to use some of the derived quantities as the main information. This technique produces formulas which are easy on the eyes, but harder to compute. Our method will be always the easiest to compute with.

4. Another natural concept is the arclength of the space curve, or the distance traveled as t moves from one value to another. If t is imagined to be the time, this is interpreted as the distance traveled. We formally define an arclength function $s(t)$ by setting

$$s(0) = 0 \text{ and } \frac{ds}{dt} = |\mathbf{r'(t)}| \text{ so that } s(t) = \int_0^t \sqrt{\mathbf{r}'(\tau) \cdot \mathbf{r}'(\tau)}\, d\tau.$$

5. It is noticed that this arclength s describes a natural coordinate system for all points of our curve and the book likes to change coordinates from t to s so our curve is parameterized in terms of s instead of t. While this works beautifully when s is a nice function of t, for most problems, the integral is hard to calculate and explicit formula for the inverse function for t in terms of s is almost impossible to find. We give three examples illustrating this.

6. Consider $\mathbf{r(t)} =< \mathbf{5\cos(t), 5\sin(t), 1} >$. This is a circle $z = 1, x^2 + y^2 = 25$. It is easy to check that $s(t) = \int_0^t \sqrt{25}\, d\tau = 5t$. So $t = \frac{s}{5}$ and we could write a new parametrization $< 5\cos(s/5), 5\sin(s/5), 1 >$.

7. Consider a similar, but little more complicated curve $\mathbf{r(t)} =< \mathbf{5\cos(t), 5\sin(t), t} >$. It is easy to check that $s(t) = \int_0^t \sqrt{25+1}\, d\tau = \sqrt{26}t$. So $t = \frac{s}{\sqrt{26}}$ and we could write a new parametrization $< 5\cos(s/\sqrt{26}), 5\sin(s/\sqrt{26}), \frac{s}{\sqrt{26}} >$.

8. Now consider a slightly faster rising helix $\mathbf{r(t)} =< \mathbf{5\cos(t), 5\sin(t), t^2} >$. It is easy to check that $s(t) = \int_0^t \sqrt{25+4t^2}\, d\tau$. The formula for $s(t)$ given by Maple is

$$1/2\, t\sqrt{25+4\,t^2} + \frac{25}{4}\, \mathit{arcsinh}\,(2/5\,t)\,.$$

Maple will also give an inverse function which is too complicated to work with (or even reproduce here).

1.2 General formulas.

We assume of space curve defined by

$\mathbf{r} =< \mathbf{f_1(t), f_2(t), f_3(t)} >$ and calculate $\mathbf{v(t)} = \mathbf{r}' =< \mathbf{f_1'(t), f_2'(t), f_3'(t)} >, \mathbf{a(t)} = \mathbf{r}'' =< \mathbf{f_1''(t), f_2''(t), f_3''(t)} >$

where we have suppressed the t from the notation of the position vector for convenience.

1. We interpret $\mathbf{v(t)}$ as the **velocity vector** and the $\mathbf{a(t)}$ as the **acceleration vector** following the usual Physics conventions. We shall also define $|\mathbf{v(t)}|$ as $\sigma(t)$ and call it the scalar speed function. We don't use the letter s since it conflicts with the arclength parameter that we also need.

2. We now have

$$\mathbf{v(t)} = \mathbf{r}' = \sigma(\mathbf{t})\mathbf{T(t)} \text{ where } \mathbf{T(t)} \text{ is always a unit vector by definition..}$$

This $\mathbf{T(t)}$ is called the unit tangent vector and serves as the unit vector on our local x-axis.

The definition has a problem if $\sigma(t) = 0$ at some point, but the curve is not smooth there and needs a special handling anyway!

3. Taking derivatives, we see that

$$\mathbf{a(t)} = \sigma'(\mathbf{t})\mathbf{T(t)} + \sigma(\mathbf{t})\frac{\mathbf{dT}}{\mathbf{dt}}.$$

Now we note a simple fact. Since $\mathbf{T(t)}$ is a unit vector, we see $\mathbf{T(t)} \cdot \mathbf{T(t)} = \mathbf{1}$, so

$$\frac{\mathbf{dT(t)}}{\mathbf{dt}} \cdot \mathbf{T(t)} + \mathbf{T(t)} \cdot \frac{\mathbf{dT(t)}}{\mathbf{dt}} = \mathbf{2}\left(\mathbf{T(t)} \cdot \frac{\mathbf{dT(t)}}{\mathbf{dt}}\right) = \mathbf{0}.$$

We define two new quantities: a scalar function $\kappa(t)$ and unit vector $\mathbf{N(t)}$ by the formula:

$$\frac{\mathbf{dT(t)}}{\mathbf{dt}} = \sigma(\mathbf{t})\kappa(\mathbf{t})\mathbf{N(t)}.$$

Of course, this defines

$$\kappa(\mathbf{t}) = \frac{\mathbf{1}}{\sigma(\mathbf{t})}\left|\frac{\mathbf{dT(t)}}{\mathbf{dt}}\right|.$$

4. Though correct, the formula is still inconvenient, since $\mathbf{T(t)} = \frac{\mathbf{1}}{\sigma(\mathbf{t})}\mathbf{r}'(\mathbf{t})$ has a complicated formula. We straighten it out once for all:

   Note that

$$\mathbf{v(t)} \times \mathbf{a(t)} = (\sigma(\mathbf{t})\mathbf{T(t)}) \times \left(\sigma'(\mathbf{t})\mathbf{T(t)} + \sigma(\mathbf{t})\frac{\mathbf{dT}}{\mathbf{dt}}\right) = (\sigma(\mathbf{t})\mathbf{T(t)}) \times \left(\sigma'(\mathbf{t})\mathbf{T(t)} + \sigma(\mathbf{t})^{\mathbf{2}}\kappa(\mathbf{t})\mathbf{N(t)}\right)$$

   and this simplifies to:

$$\mathbf{v(t)} \times \mathbf{a(t)} = \sigma(\mathbf{t})\mathbf{T(t)} \times \sigma(\mathbf{t})^{\mathbf{2}}\kappa(\mathbf{t})\mathbf{N(t)} = \kappa(\mathbf{t})\sigma(\mathbf{t})^{\mathbf{3}}\mathbf{T(t)} \times \mathbf{N(t)}.$$

   Thus we get the formula:

$$\kappa(t) = \frac{|\mathbf{v(t)} \times \mathbf{a(t)}|}{\sigma(t)^3}.$$

   A special case of this formula can be worked out when we have a plane curve given by a parametrization

$$\mathbf{r(t)} =< \mathbf{t}, \mathbf{f(t)}, \mathbf{0} > .$$

   Then we get:

$$\mathbf{v(t)} =< \mathbf{1}, \mathbf{f}'(\mathbf{t}), \mathbf{0} >, \mathbf{a(t)} =< \mathbf{0}, \mathbf{f}''(\mathbf{t}), \mathbf{0} >$$

   and it is easy to calculate:

$$\kappa(t) = \frac{|f''(t)|}{(1 + f'(t))^{3/2}}.$$

5. **Calculation strategy**

   Finally, we indicate the best sequence of operations to enable determination of necessary quantities.

   - Start with the given $\mathbf{r(t)}$ and then calculate:

     (1) $$\mathbf{r}'(\mathbf{t}) = \mathbf{v(t)} \text{ and } \mathbf{r}''(\mathbf{t}) = \mathbf{a(t)}$$

   - Next calculate

     (2) $$v(t) \times a(t) \text{ and call it } H(t) \text{ for now.}$$

   - Now, you have:

$$\sigma(t) = |v(t)| \text{ also called the speed, if this is the description of a motion.}$$

     Also,

     (3) $$T = \frac{v(t)}{\sigma(t)}.$$

   - Now,

     (4) $$\frac{|H(t)|}{\sigma^3(t)} = \kappa(t)$$

     Also

     (5) $$B(t) = \frac{H(t)}{|H(t)|}.$$

- Now

$$B(t) \times T(t) = N(t). \tag{6}$$

- You can also find other pieces of the formula, if needed.

$$\sigma'(t) = \frac{v(t) \cdot a(t)}{\sigma(t)}. \tag{7}$$

To be continued ...