1   Integration in many variables.

1.1   Basic Definition.

The integration in one variable was developed along these lines:

1. $\int_I f(x)\,dx$, where I is any interval on the real line was defined to be a limit of certain Riemann sums defined thus.

   - By a partition P of I, we mean any sequence of disjoint subintervals $I_1, I_2, \cdots, I_n$ whose union is I. By a sample sequence S, we mean a corresponding sequence of values $f(c_1), f(c_2), \cdots, f(c_n)$ where $c_i \in I_i$ for each $i = 1, \cdots n$.

   - Then an associated Riemann sum is

   $$R(P, S, f) = \sum_{i=1}^{n} f(c_i)|I_i| \text{ where } |I_i| \text{ is the length of } I_i.$$

   - By the norm of P, we mean $|P| = \max\{|I_i| \mid i = 1, \cdots n\}$.
   - Finally, we define:
   $$\int_I f(x)\,dx = \lim_{|P| \to 0} R(P, S, f)$$

   - The function f is said to integrable on I if the above limit exists.

2. The above definition can be readily extended to any number of variables. We illustrate the case of two variables.

   We define
   $$\int_D f(x, y)\,dx\,dy = \lim_{||P|| \to 0} R(P, S, f)$$

   where

   - P is now a partition of the domain D into disjoint pieces $\Delta_1, \Delta_2, \cdots, \Delta_n$.
   - S is any sample sequence $f(c_1, d_1), f(c_2, d_2), \cdots, f(c_n, d_n)$ where $(c_i, d_i) \in \Delta_i$.
   -
   $$R(P, S, f) = \sum_{i=1}^{n} f(c_i, d_i)|\Delta_i| \text{ where } |\Delta_i| \text{ is the area of } \Delta_i.$$

   - As before, $|P| = \max\{|\Delta_i| \mid i = 1, \cdots n\}$.

3. A function f defined on a domain D is said to be integrable if $\int_D f\,dX$ exists. Here we have used X is intended to denote a single variable x, or two variables (x, y) or more.

   Riemann's theorem says that a continuous function is integrable when D is a nice closed bounded region (a box-like set) - an interval, a rectangle, or a higher dimensional analog. Usually, we are interested in a closed region which is contained in such "boxes" and the theorem still holds if its boundary is reasonably smooth. (Details are left to a higher level course).

Further generalizations include when the function is even allowed many discontinuities, provided that they form a small set in a technical sense. Extensions to unbounded domains are also made by a limiting process.

We sketch a brief argument for a continuous function on a box-like domain.

For any partition $P = \{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ we define two special sample sequences:

U where we choose a sample point which has a maximum value in its piece and

L which has a minimum value in its piece.

Then clearly $R(P, L, f) \leq R(P, S, f) \leq R(P, U, f)$ for any sample sequences S for the partition P.

Let the area of our domain be Δ and for any given ϵ choose a δ so that whenever $|X - Y| < \delta$ we have $|f(X) - f(Y)| < \frac{\epsilon}{\Delta}$. The existence of such a δ comes from a careful analysis of continuity (or more specifically the so-called uniform continuity).

The special sequences L and U exist for a continuous function and it is possible to argue that if $|P|$ is less than δ then

$$R(P, U, f) - R(P, L, f) \leq \frac{\epsilon}{\Delta} \left(\Delta_1 + \Delta_2 + \cdots + \Delta_n\right) = \epsilon.$$

Letting ϵ go to zero, we see that $R(P, U, f)$ and $R(P, L, f)$ both approach the same limit and all other sample sequence must also give the same limit since they are sandwiched between the two!

This proves the existence of the integral.

4. Many evident properties of the integral can be deduced from this definition.

We shall use a general notation $\int_D f \, dX$ to denote a multivariate integral, where $f = f(X) = f(x_1, \cdots, x_n)$ and $dX = dx_1 \, dx_2 \cdots dx_n$. The region D will be in n-space.

Of course, our concern is for 1 or 2 or 3 variables.

Linearity $\int_D (af + bg) \, dX = a \int_D f \, dX + b \int_D g \, dX$, where a, b are constants.

Additivity of domains If D and E are domains which are disjoint except along some common curves, then $\int_{D \cup E} f \, dX = \int_D f \, dX + \int_E f \, dX$. Here, the common curves, if any, should not be too "kinky".

Domination If $f(X) \geq g(X)$ for all X in D, then $\int_D f \, dX \geq \int_D g \, dX$. This is often useful to estimate integrals which are difficult to evaluate.

Basic value. If we integrate the constant function 1, then we get $\int_D dX$ to be the area of D. The word area is replaced by length in dimension 1 and by volume in dimension 3. In fact, mathematically, this can be taken to be a **definition** of the length/area/volume in the respective cases.

In higher mathematics, by modifying the dX to $g \, dX$ for a chosen function g, people define the concepts for different metrics based on the chosen geometry.

In Statistics, a certain probability density function σ is used to give the probability associated with D as $\int_D \sigma \, dX$.

In Physics, we use the density function ρ to calculate mass.

1.2 Calculation Techniques for Double Integrals.

1. **Iterated Integrals.**

   We discuss special case of two variables here. Three variables are handled similarly.

   Suppose we are interested in $I = \iint_R f(x,y)\, dx\, dy$ and our region D can be described in the following way:

   Type I domain:

   - All points of R are contained between vertical lines $x = a$ and $x = b$ with $a < b$.
   - Moreover, for each $x \in [a,b]$ the points of R can be described as inside the interval $[(x, l(x)), (x, u(x))]$.

   Then we can calculate

   $$I = \int_{x=a}^{b} \left(\int_{y=l(x)}^{u(x)} f(x,y)\, dy \right) dx.$$

   Here, the inside integral is considered as an integral in y with x treated as a constant.

   This can be described as integral by vertical sections or x-sections.

   Type II domain:

   - All points of R are contained between horizontal lines $y = c$ and $y = d$ with $c < d$.
   - Moreover for each $y \in [c,d]$ the points of R can be described as inside the interval $[(l(y), y), (u(y), y]$.

   Clearly, there is another possibility

   $$I = \int_{y=c}^{d} \left(\int_{x=l(y)}^{u(y)} f(x,y)\, dx \right) dy.$$

   This can be described as integral by horizontal sections or y-sections.

2. **Fubini's Theorem.**

   The important Fubini theorem assures us that both these are equal and correct, if the function is continuous.

   Comment We note that for a random domain R, the sections may not come out as single intervals, but may break up into several pieces. We can still calculate by splitting the integral over nice domains and using additivity. The nice domains are those for which the sections come out as intervals, i.e. are of types I or II.

3. **An example.** Consider the integral

   $$I = \int_{y=0}^{4} \int_{x=0}^{\sqrt{y}} xy\, dx\, dy.$$

   The region of integration is bounded by the lines $x = 0, y = 4, y = x^2$.

   The given integral is clearly set up as a type II integral.

   First, we evaluate

   $$I = \int_{y=0}^{4} \left(x^2 y/2 \right) \Big|_{x=0}^{\sqrt{y}} dy = \int_{y=0}^{4} \left(y^2/2 \right)\, dy = (y^3/6) \Big|_{y=0}^{4} = 64/3.$$

3

This region can be worked out as type I by setting

$x \in [0, 2]$ and for any such x, the points of the region are in the interval $[(x, 0), (x, x^2)]$.

The reader should check that it gives the same answer.

4. A variant example. If we change the above example to

$$\int_{y=0}^{4} \int_{x=y/4}^{\sqrt{y}} xy \, dx \, dy$$

then the region is bounded by $y = 4x, y = 4, y = x^2$ and conversion to type I will require two integrals, namely:

$$\int_{x=0}^{1} \int_{y=x^2}^{4x} xy \, dy \, dx + \int_{x=1}^{2} \int_{y=x^2}^{4} xy \, dy \, dx.$$

This requires a careful sketch!

5. **Change of variables.** Suppose we have to evaluate an integral $\iint_R f(x, y) \, dx \, dy$ over some region R.

Sometimes the process can be simplified by a substitution $x = p(u, v), y = q(u, v)$. As in the one variable case, this requires transformation of the integrand as well as the domain of integration.

- We assume that the change of variables $(x, y) \to (u, v)$ transforms the domain R into a domain S in the (u, v) plane in a one to one fashion.[1]

- The differential $dA = dx \, dy$ is transformed to

$$J_{u,v}(p(u, v), q(u, v)) du \, dv = \left\| \begin{matrix} p_u & p_v \\ q_u & q_v \end{matrix} \right\| du \, dv = |p_u q_v - p_v q_u| du \, dv.$$

  This $J_{u,v}(p(u, v), q(u, v))$ (or briefly $J_{u,v}(x, y)$ is called the Jacobian and can be thought of as a measure of the limit of the ratio of the transformed area of a rectangle of sides $\Delta(x), \Delta(y)$ and the original area $\delta(u) \cdot \delta(v)$. Another explanation is given below.

- The region of integration R is changed to a region S in the (u, v)- plane such that the map $(u, v) \to (x, y) = (p(u, v), q(u, v))$ is one to one (at least in outside the boundary.

- Many times, we know the inverse functions $u = g(x, y), v = h(x, y)$. In this case, we may first compute the $J_{x,y}(u, v) = J_{x,y}(g(x, y), h(x, y))$ and it can be shown that $J_{u,v}(x, y) = 1/J_{x,y}(u, v)$.

6. **Example of Change of variables.**

Consider the problem of finding $\iint_R xy \, dA$ where

$$R = \{(x, y) \mid 0 \le x + y \le 1, \ 0 \le y - 2x \le 1\}$$

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[1]Technically, this presumes that we can solve (u, v) in terms of (x, y) on our domain. However, this may be hard to do, for complicated substitutions. Often non zeroness of the jacobian on the region except possibly on the boundary, is a good indication.

It is a parallelogram with corners $(0,0), (1/3, 2/3), (0, 1), (-2/3, 1/3)$. It is easy to check that this is neither type I nor type II, unless we split up the integral in two pieces.

But we make a substitution thus:

Set $u = x + y, v = -2x + y$. then our region seems to transform to a nice square S, described by: $0 \leq u \leq 1$ and $0 \leq v \leq 1$. We can solve the equations for x, y as

$$x = \frac{u - v}{3} \text{ and } y = \frac{u + 2v}{3}.$$

Thus,

$$J_{x,y}(u, v) = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3, \text{ and } J_{u,v}(x, y) = \begin{vmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{3}{9} = \frac{1}{3}.$$

Obviously, their product is 1 as stipulated. Thus, our integral now transforms as:

$$\iint_S \frac{(u - v)}{3} \frac{(u + 2v)}{3} \frac{1}{3} du\, dv = \int_{v=0}^1 \int_{u=0}^1 \frac{(u - v)}{3} \frac{(u + 2v)}{3} \frac{1}{3} du\, dv.$$

This last integral is readily shown to be $\frac{-1}{108}$.

7. **The Polar Substitution.** One of the most useful substitution for evaluating double integrals is the polar coordinates.

Thus, we have:

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos(\theta), r\sin(\theta))|r|\, dr\, d\theta$$

where the region S is in the (r, θ) plane so that the map $(x, y) = (r\cos(\theta), r\sin(\theta))$ maps S to R in a one to one fashion.

The quantity $r\, dr\, d\theta$ is the transformation of $dx\, dy$ as explained above, since $J_{r,\theta}(r\cos(\theta), r\sin(\theta)) = r$. Usually r is taken to be non negative and so $|r|$ may be replaced by r.

For example, if $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$- the solid disc, then the integral

$$V = \iint_R \sqrt{1 - (x^2 + y^2)}\, dx\, dy$$

gives the volume of the hemi (half) sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane.

In polar coordinates, this becomes

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^1 \sqrt{1 - r^2}\, r\, dr\, d\theta.$$

By a simple substitution:

$$\int \sqrt{1 - r^2}\, r\, dr = \frac{(1 - r^2)^{3/2}}{3/2}(-1/2) = -\frac{(1 - r^2)^{3/2}}{3/2}.$$

Hence

$$V = \int_{\theta=0}^{2\pi} \left(-\frac{-(1 - 1^2)^{3/2}}{3} + \frac{(1 - 0^2)^{3/2}}{3} \right) d\theta = \int_{\theta=0}^{2\pi} \frac{1}{3} d\theta = \frac{2\pi}{3}$$

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8. **Another explanation of the Jacobian.**

The formula for the relation between $du\,dv$ and $dx\,dy$ can be described by considering these differentials as vectors, rather than scalars. Thus, we replace dx by $\overrightarrow{dx}$ a vector in the x-direction. Similarly, dy is replaced by a vector in the y-direction.

Now, $u = p(x, y), v = q(x, y)$ result in:

$$\overrightarrow{du} = p_x\overrightarrow{dx} + p_y\overrightarrow{dy} \text{ and } \overrightarrow{dv} = q_x\overrightarrow{dx} + q_y\overrightarrow{dy}.$$

Further, we interpret $dx\,dy$ as the magnitude of $\overrightarrow{dx} \times \overrightarrow{dy}$. Thus, we calculate:

$$
\begin{aligned}
\overrightarrow{du} \times \overrightarrow{dv} &= \left(p_x\overrightarrow{dx} + p_y\overrightarrow{dy}\right) \times \left(q_x\overrightarrow{dx} + q_y\overrightarrow{dy}\right) \\
&= p_x q_x\overrightarrow{dx} \times \overrightarrow{dx} + p_y q_y\overrightarrow{dy} \times \overrightarrow{dy} + p_x q_y\overrightarrow{dx} \times \overrightarrow{dy} + p_y q_x\overrightarrow{dy} \times \overrightarrow{dx} \\
&= 0 + 0 + (p_x q_y - p_y q_x)\,\overrightarrow{dx} \times \overrightarrow{dy}
\end{aligned}
$$

This gives the necessary formula for the Jacobian $J_{x,y}(u, v)$.

9. **Formula for the surface area.** The above explanation also gives a nice formula to compute the area of a parametric surface given by vector equations:

$$\mathbf{r(u, v)} =< \mathbf{f(u, v), g(u, v), h(u, v)} > .$$

We naturally construct the derivative:

$$\mathbf{dr} =< \mathbf{f_u, g_u, h_u} > \mathbf{du}+ < \mathbf{f_v, g_v, h_v} > \mathbf{dv} = \mathbf{r_u du} + \mathbf{r_v dv}.$$

It is possible to see that vectors $\mathbf{r_u}$ and $\mathbf{r_v}$ are tangent vectors for our surface and $\overrightarrow{du} \times \overrightarrow{dv}$ produces an images $\mathbf{r_u du} \times \mathbf{r_v dv}$ on the tangent plane.

This will be discussed in detail later. However, we deduce a formula for the surface area of the surface as u, v move in a region R:

$$\text{Surface area} = \iint_R |\mathbf{r_u} \times \mathbf{r_v}|\mathbf{du\,dv}.$$

For instance, for our hemisphere described above, the surface $x^2 + y^2 + z^2 = 1$ is parameterized as

$$\mathbf{r} =< \mathbf{x, y,} \sqrt{\mathbf{1 - (x^2 + y^2)}} > .$$

Using x, y as parameters, we get:

$$\mathbf{r_x} =< \mathbf{1, 0,} -\frac{\mathbf{x}}{\sqrt{\mathbf{1 - (x^2 + y^2)}}} >, \quad \mathbf{r_y} =< \mathbf{0, 1,} -\frac{\mathbf{y}}{\sqrt{\mathbf{1 - (x^2 + y^2)}}} >$$

and hence

$$\mathbf{r_x} \times \mathbf{r_y} =< \frac{\mathbf{x}}{\sqrt{\mathbf{1 - (x^2 + y^2)}}}, \frac{\mathbf{y}}{\sqrt{\mathbf{1 - (x^2 + y^2)}}}, \mathbf{1} > .$$

This has magnitude

$$\sqrt{\left(\frac{x}{\sqrt{1-(x^2+y^2)}}\right)^2 + \left(\frac{y}{\sqrt{1-(x^2+y^2)}}\right)^2 + 1} = \sqrt{\frac{1}{1-(x^2+y^2)}} = \frac{1}{\sqrt{1-(x^2+y^2)}}.$$

Thus, our surface area of the hemisphere is equal to

$$\iint_R \frac{1}{\sqrt{1-(x^2+y^2)}}\,dx\,dy = \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta.$$

Here, we have used the polar substitution as before. The evaluation is easily worked out to 2π.

10. Formula for a Special surface. Often our surface is given by

$$\mathbf{r}(\mathbf{x},\mathbf{y}) = <\mathbf{x},\mathbf{y},\mathbf{f}(\mathbf{x},\mathbf{y})> \quad \text{as } (x,y) \text{ vary over } R.$$

In this case, we can simply record the formula:

$$\text{Surface area} = \iint_R \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy.$$

As seen in the above example, we may need algebraic manipulation or additional substitutions to evaluate.

2   Applications of Double Integrals.

If R is a domain in the plane, then the double integral $I = \iint_R f\,dA$ is used in many application by choosing suitable functions f. We list these below.

1. **Volume of a solid above R.** If $f(x,y)$ is the length of the section of a solid above R, then I gives its volume.

2. **Mass of R.** If we take $f(x,y)$ to be the density at (x,y) then I gives the mass of the laminar region R.

3. **Center of Mass.** The center of mass $(\overline{x}, \overline{y})$ of a laminar region R with density function $\rho(x,y)$ is found thus:
$$\overline{x} = \frac{\iint_R x\rho(x,y)\,dA}{m}, \quad \overline{y} = \frac{\iint_R y\rho(x,y)\,dA}{m}$$
where m is the mass as found above.

4. **Moments.** The moment of a laminar region about an axis in the plane is obtained by integrating the "directed" distance from the axis multiplied by the density function.

   Thus, **for example** the moment about the x-axis is given by

$$\iint_R y\rho(x,y)\,dA.$$

5. **Moments of Inertia** The moment of inertia about an axis in the plane is obtained by integrating the square of the distance from the axis multiplied by the the density function.

Thus, **for example** the moment about the x-axis is given by

$$\iint_R y^2 \rho(x,y)\, dA.$$

This leads to the radius of gyration about the same axis by the formula

$$\text{radius} = \sqrt{\frac{\text{Moment of Inertia about the axis}}{\text{mass}}}.$$

In this context the axis may be perpendicular to the plane. We simply integrate the square of the distance from the point in the plane on the axis. It may be called the moment of inertia about the point.

Thus, the moment of inertia about the origin is

$$\iint_R (x^2 + y^2)\rho(x,y)\, dA.$$

6. **Probability** If $p(x,y)$ denotes a probability distribution, then the probability of a random point (x,y) being in R is given by $\iint_R p(x,y)\, dA$.

7. **Expected value.** If we have some real function $f(x,y)$ and $p(x,y)$ the probability distribution as above, then $\int_R f(x,y)p(x,y)\, dA$ is called the expected value of $f(x,y)$ on R.

3 Triple Integrals.

We do not make explicit notes here, but triple integrals are handled by a simple generalization of the above.

1. The Riemann sums are defined by using partitions of a box into small boxes.The same Riemann Theorem holds.

2. If we integrate over a domain R then we make iterated integrals just as before.

   For example consider the integral of $f(x,y,z) = x+y+z$ over the region in the first octant below the lplane $x/2 + y/3 + z/5 = 1$. The region is a tetrahedron with vertices $O = (0,0,0), A = (2,0,0), B = (0,3,0), C = (0,0,5)$. If we note that the z-coordinates vary from 0 to 5 in R, then we simply write

   $$\int \int \int_R f(x,y,z)dv = \int_{z=0}^{z=5} \left(\int \int_D f(x,y)dxdy \right) dz$$

   For a fixed z, the inside integral is over a regio bounded by x,y axes and the line $x/2 + y/3 = 1 - z/5$. Note that z is treated as fixed here.

   You can check that the inside integral becomes

   $$\int_{y=0}^{3(1-z/5)} \int_{x=0}^{2(1-y/3-z/5)} f(x,y)dxdy.$$

   You should check that the whole integral evaluates to $25/2$.

   More examples will be done in class and in the homework.

4   Surface Integrals.

If a surface S is given in parametric form $r = <x(u,v), y(u,v), z(u,v)>$ where $(u,v) \in D$, we define the integral of a function $f(x,y,z)$ on S by the formula:

$$\iint_S f(x,y,z)\, dS = \iint_D f(r(u,v))|r_u \times r_v|\, dA.$$

However, surfaces don't usually come with a ready parametrization and it is useful to have a more general formula handy.

Let the surface S be described by an equation $g(x,y,z) = 0$, so that we have the basic relation $g_x dx + g_y dy + g_z dz = 0$.

It is possible to think of x, y as parameters at a point where $g_z \neq 0$. In this case, taking $u = x, v = y$, we see that

$$r_u \times r_v = <1, 0, -\frac{g_x}{g_z}> \times <0, 1, -\frac{g_y}{g_z}> = <\frac{g_x}{g_z}, \frac{g_y}{g_z}, 1> = \frac{1}{g_z}\nabla(g).$$

Thus, our integral reduces to

$$\iint_S f(x,y,z)\, dS = \iint_D f|\nabla(g)|\frac{dx\, dy}{|g_z|}.$$

It is easy to see that a similar formula holds when $g_y \neq 0$ and we take x, z as parameters and when $g_x \neq 0$ we take y, z as parameters.

We define the **Fundamental differential** on S to be:

$$\omega_g = \frac{dx\, dy}{g_z} = \frac{dy\, dz}{g_x} = \frac{dz\, dx}{g_y}.$$

Now we have a single formula for all cases:

$$\iint_S f(x,y,z)\, dS = \iint_D f|\nabla(g)|\,|\omega_g|.$$

Orientable Surface. We note that $\nabla(g)$ defines a normal to our surface at all smooth points (points where $\nabla(g)$ is defined and non zero.) Also, it varies continuously if g is continuous. Then such a surface has a well defined unit normal vector field at all smooth points, namely $n = \frac{\nabla(g)}{|\nabla(g)|}$.

If our surface encloses a bounded solid (like a sphere, then we can even make sense out of an outward or inward normal.

We now define the surface integral of a vector field F on the surface S by defining it as the integral of the function $F \cdot n = F \cdot \frac{\nabla(g)}{|\nabla(g)|}$. Our formula becomes

$$\iint_S F \cdot n|\nabla(g)|\,|\omega_g| = \iint_D F \cdot \nabla(g)\,|\omega_g|.$$

For a parametric surface, the formula becomes:

$$\iint_D f(r(u,v)) \cdot r_u \times r_v\, dA.$$

Note that the above formula presumes a certain direction of the unit normal as determined by $\nabla(g)$ or $r_u \times r_v$. We multiply by -1 if we wish to change the direction.

To be continued ...