1 Line Integrals in Plane.

1.1 Introduction.

1.1.1 Curves.

A piece of a smooth curve C is assumed to be given by a vector valued position function $P(t)$ (or $\mathbf{r}(\mathbf{t})$) as the parameter t moves in an interval I on the real line. **The smoothness** means the function is differentiable and the derivative $P'(t) = v(t) = \mathbf{r}'(\mathbf{t})$ is non zero on I.

If I is closed, then the curve has a well defined starting point and an end point.

A curve is said to be **simple** if different values of t give different points $P(t)$, except the beginning point is allowed to be the same as the end point. For example, consider the curve:

$$P(t) =< \cos(t), \sin(t) > \text{ for } t \in [a, b].$$

Depending on the values of a, b, this curve traces parts of the unit circle and has the starting and end points as $P(a)$ and $P(b)$.

If $[a, b] = [0, \pi]$, then we get the upper semicircle. If $[a, b] = [0, 2\pi]$ then we get the complete circle S where the starting and end point are the same. This is an example of a simple closed curve, where each value of t gives a different point of the curve, except for the start and end.

If we take $[a, b] = [0, 4\pi]$, then this is not a simple curve and each point is visited twice!

The curve given by the formula $P(t) =< \cos(2\pi - t), \sin(2\pi - t) >$ describes the same set as S, but traced in reverse. We denote such a curve by $-S$ or curve S traced backwards.

A curve in general is composed by smooth pieces joined at one point. Given two curves C_1, C_2 such that the end point of C_1 is equal to the start of C_2, we define the curve $C_1 + C_2$ as the curve obtained by tracing C_1 followed by C_2. We can add several pieces as desired.

By $C - C$ we mean $C + (-C)$ i.e. the travel on C followed by a return trip!

Given a piece of curve, we may describe it by a different parametrization and we will consider it as the same piece of the curve if the points are in one to one correspondence. [1]

For example, our circle S is also described by the parametrization $P(2t)$ where $t \in [0, \pi]$.

1.1.2 Integral on curves.

Given a piece of a plane curve $C : r(t) =< u(t), v(t) >$ with $t \in [a, b]$ and a function $f(x, y)$, we define **integral of $\mathbf{f(x, y)}$ on C to be:**

$$\int_C f(x, y)ds = \int_a^b f(u(t), v(t))\sqrt{u'(t)^2 + v'(t)^2}dt.$$

It is not hard to show that this integral is independent of the choice of parametrization.

If we have a vector field $F(x, y) =< F_1(x, y), F_2(x, y) >$, then **integral of $\mathbf{F(x, y)}$ on C to be:**

$$\int_C F \cdot T ds = \int_C F(u(t), v(t)) \cdot T(t)\sqrt{u'(t)^2 + v'(t)^2}dt.$$

[1] If our parametrization runs through our geometric curve multiple times, then the new parametrization has to run the same number of times. Usually, we avoid such technicalities and use simple curves.

If we recall that $T(t) = \frac{<u'(t),v'(t)>}{\sqrt{u'(t)^2+v'(t)^2 dt}}$, then it is easy to see that

$$\int_C F \cdot T ds = \int_C (F_1(u(t), v(t))u'(t) + F_2(u(t), v(t))v'(t))\, dt.$$

This is clearly equal to $\int_C F \cdot dr$ and independent of parametrization.

More generally, we can define the integral of any differential $P(x, y)dx + Q(x, y)dy$ (also briefly written as $Pdx + Qdy$) on a parametric curve C by

$$\int_C < P, Q > \cdot < dx, dy >= \int_C Pdx + Qdy = \int_a^b (P(u(t), v(t))u'(t) + Q(u(t), v(t))v'(t))\, dt.$$

For a space curve, there are obvious generalizations:

$$\int_C f(x, y, z)ds = \int_a^b f(u(t), v(t), w(t))\sqrt{u'(t)^2 + v'(t)^2 + w'(t)^2} dt$$

$$\int_C Pdx+Qdy+Rdz = \int_a^b (P(u(t), v(t), w(t))u'(t) + Q(u(t), v(t), w(t))v'(t) + R(u(t), v(t), w(t))w'(t))\, dt.$$

and

$$\int_C F \cdot T ds = \int_C (F_1(u(t), v(t), w(t))u'(t) + F_2(u(t), v(t), w(t))v'(t) + F_3(u(t), v(t), w(t))w'(t))\, dt.$$

1.2 Fundamental Theorem

If we have a vector field $F = \nabla(f)$ for some function f, then we see that

$$\int_C F \cdot dr = \int_a^b (f_x(u(t), v(t))u'(t) + f_y(u(t), v(t))v'(t))\, dt = \int_a^b d(f(u(t), v(t)) = f(u(b), v(b)) - f(u(a), v(a)).$$

Thus, the integral depends only on the values of $f(x, y)$ at the two endpoints of C. In particular, we get the same integral if we replace the curve by any other curve which has the same starting and ending point.

We define vector fields of the form $\nabla(f)$ to be **conservative**.

If F denotes the force field acting on a particle moving on a curve C from $r(a)$ to $r(b)$, then the work done is known to be given by $\int_C F \cdot dr$. Thus, if $F = \nabla(f)$, then the work does not depend on the path and evaluates to $f(r(b)) - f(r(a))$. This is the reason for the term "conservative". The function f is called the potential function of the field.

Given a field $< P, Q >$, when is it conservative? If P, Q have continuous derivatives in a domain, then by Clairaut's theorem, we get $P_y = f_{xy} = f_{yx} = Q_x$ as a necessary condition.

This may not be sufficient unless our domain of definition is nice. The technical term is "simply connected" and roughly means it has no "holes".

A standard example of a non conservative field which satisfies the necessary condition is given by

$$F =< P, Q >=< \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} > .$$

It can be readily shown that $\int_C F \cdot dr = 2\pi$ if we let C the unit circle $r(t) =< \cos(t), \sin(t) >$ as t varies in $[0, 2\pi]$. Since the answer is not zero even though the starting and ending points are the same, it is not conservative. The problem is that the vector field is not defined at the origin which is enclosed by the curve.

1.3 Green's Theorem

One of the most important and useful theorem about line integrals is:

Green's Theorem Suppose that C is a piecewise smooth, simple closed curved traced counter-clockwise (positively oriented) and D is the enclosed region. If the functions P, Q have continuous partial derivatives on an open region containing D, then

$$\int_C Pdx + Qdy = \int\int_D (Q_x - P_y)\, dA.$$

Comments:

- One way to formally define positive orientation is to require that the unit normal N points towards the region at all points (where it exists).
- Note that continuous differentiability is assumed in a slightly bigger region than D itself. Thus, the functions have to be well behaved in D as well as in a neighborhood of C.
- Sometimes the integral around the positively oriented C is denoted as $\oint_C$.
- It is possible to define a formalism whereby the oriented boundary C can be thought of as a derivative of the region ∂D. Moreover, it is possible to set up a formalism of derivatives of differentials which give
$$d(Pdx + Qdy) = -P_y\, dx\, dy + Q_x\, dx\, dy.$$
Then the theorem can be reworded as
$$\int\int_D d(Pdx + Qdy) = \int_{\partial D} Pdx + Qdy.$$
The fundamental theorem of Calculus can also be rephrased in the same language
$$\int_I d(f) = \int_{\partial I} f = f(b) - f(a)$$
where we interpret the oriented boundary of an interval $I = [a, b]$ as $[b] - [a]$.

  Later on we will see two more theorems with the same philosophy.

1.3.1 Uses of Green's Theorem.

Many results can be deduced using the theorem.

1. If $Pdx + Qdy$ is closed, meaning $Q_x - P_y = 0$, then $\int_C Pdx + Qdy = 0$. Thus, it proves the result about the conservative vector fields without finding the potential function.
2. $$\int_C -ydx = \int_C xdy = \frac{1}{2}\int_C (xdy - ydx) = \text{ the area of the enclosed region } D$$
For example the area inside the ellipse $r(t) =< a\cos(t), b\sin(t) >$ is easily found by
$$\frac{1}{2}\int_C (xdy - ydx) = \frac{1}{2}\int_C a\cos(t)(b\cos(t) - b\sin(t)(-a\sin(t))\, dt = \frac{1}{2}\int_0^{2\pi} abdt = \pi ab.$$
Note how the third formula made the work easier!

3. Changing the path without changing the integral. Suppose that we have $P_y = Q_x$ in a region D around the origin. Then for any simple closed positively oriented piecewise smooth curve C going around the origin the integral $\int_C P dx + Q dy$ is the same.

   The idea of the proof is to consider some small circle totally contained inside C and apply the Green's theorem to the region between the two curves, using a mild generalization of the theorem.

2 Line Integrals in Space.

If we consider a curve in space lying over a surface, then it can enclose a region of the surface. We now set up notation for line integrals on such curves and get a generalization of Green's Theorem known as Stokes' Theorem.

2.1 Curl and Div.

We wish to know whether a three dimensional vector field $F =< P, Q, R >$ is conservative, i.e. $F = \nabla(f)$ for some f.

We define a vector operation $\text{curl } F =< R_y - Q_z, P_z - R_x, Q_x - P_y >$. We note that this can be formally understood as a cross product $\nabla \times F$, where ∇ is interpreted as a vector operator $< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$.

It is readily checked that $\text{curl } \nabla(f) = 0$ by Clairaut's theorem in three dimensions.

Thus, $\text{curl } F = 0$ is a necessary condition for F to be a conservative field. In general, it may not be sufficient, unless our field F is defined with continuous partial derivatives in a simply connected region.

Another useful function is divergence, defined for a vector field $F =< P, Q, R >$ as $\text{div } F = \nabla \cdot F = P_x + Q_y + R_z$. This is called the divergence of the field F.

It is easily checked (assuming continuous partial derivatives) that $\nabla \cdot \nabla \times F = \text{div } \text{curl } F = 0$. Thus, $\text{div } G = 0$ is a necessary condition for a field G to be of the form $\text{curl } F$. [2]

The meaning of the Curl and Div If $F =< P, Q, R >$ describes the vector field of the velocity of a fluid flow, then the field $\text{curl } F$ denotes the axis of rotation of the fluid at a given point and the scalar quantity $\text{div } F$ measures the tendency to move away from (diverge) the point.

This idea can be visualized by considering the simplest examples of F.

- If $F =< x, 0, 0 >$ then it is easy to see that the flow from any point (a, b, c) is strictly along the x-axis, in the positive or negative direction depending on the sign of a. The fact that $\text{div } F = 1$ expresses this nature. If we fix a little sphere around a point, then it can be argued by a suitable integration that there is net fluid movement in the positive x-direction. $\text{curl } F = 0$ and there is no rotation.

- If $F =< y, 0, 0 >$, then $\text{curl } F =< 0, 0, -1 >$ suggesting a rotation in a plane parallel to the xy plane in the clockwise direction. This rotation is caused by the fluid moving faster with the increasing value of y. There is a simple way to measure this. Choose a clockwise path around a point, say $(0, 1, 0)$ in this plane by $r(t) =< \cos(t), 1 - \sin(t), 0 >$ as t goes from 0 to 2π. Integrate our field along this path, i.e. compute

$$\int_0^{2\pi} F \cdot dr = \int_0^{2\pi} (1 - \sin(t))(-\sin(t))dt = \pi.$$

[2]As before, we may ask for the converse, but it leads to more complicated notions of contractible spaces. This is beyond the scope of our course.

This integral is called the circulation of the fluid in our field. Along a counterclockwise field, this would have come out $-\pi$ and this can be shown to be the value of the curl F times the area enclosed by our path. The divergence div $F = 0$, as the net fluid movement across a little sphere is zero across all lines parallel to the x-axis.

- A similar analysis can be done for $F =< z, 0, 0 >$ to get curl $F =< 0, 1, 0 >$ indicating a rotation about the y-axis. The divergence is zero for a similar reason.

Another useful function is the Laplacian ($\nabla \cdot \nabla$ or shortened to ∇^2) which is defined as

$$\text{div } (\nabla(f)) = \nabla^2(f) = \nabla\cdot < f_x, f_y, f_z >= f_{xx} + f_{yy} + f_{zz}.$$

Even though we have used a three variable definition, the definition naturally makes sense for any number of variables.

A function f is said to be harmonic if $\nabla^2(f) = 0$.

3 Surface Integrals.

If a surface S is given in parametric form $r =< x(u, v), y(u, v), z(u, v) >$ where $(u, v) \in D$, we define the integral of a function $f(x, y, z)$ on S by the formula:

$$\int\int_S f(x, y, z)\, dS = \int\int_D f(r(u, v))|r_u \times r_v|\, dA.$$

However, surfaces don't usually come with a ready parametrization and it is useful to have a more general formula handy.

Let the surface S be described by an equation $g(x, y, z) = 0$, so that we have the basic relation $g_x dx + g_y dy + g_z dz = 0$.

It is possible to think of x, y as parameters at a point where $g_z \neq 0$. In this case, taking $u = x, v = y$, we see that

$$r_u \times r_v =< 1, 0, -\frac{g_x}{g_z} > \times < 0, 1, -\frac{g_y}{g_z} >=< \frac{g_x}{g_z}, \frac{g_y}{g_z}, 1 >= \frac{1}{g_z}\nabla(g).$$

Thus, our integral reduces to

$$\int\int_S f(x, y, z)\, dS = \int\int_D f|\nabla(g)|\, \frac{dx\, dy}{|g_z|}.$$

It is easy to see that a similar formula holds when $g_y \neq 0$ and we take x, z as parameters and when $g_x \neq 0$ we take y, z as parameters.

We define the **Fundamental differential** on S to be:

$$\omega_g = \frac{dx\, dy}{g_z} = \frac{dy\, dz}{g_x} = \frac{dz\, dx}{g_y}.$$

Now we have a single formula for all cases:

$$\int\int_S f(x, y, z)\, dS = \int\int_D f|\nabla(g)|\, |\omega_g|.$$

Orientable Surface. We note that $\nabla(g)$ defines a normal to our surface at all smooth points (points where $\nabla(g)$ is defined and non zero.) Also, it varies continuously if g is continuous. Then such a surface has a well defined unit normal vector field at all smooth points, namely $n = \frac{\nabla(g)}{|\nabla(g)|}$.

If our surface encloses a bounded solid (like a sphere, then we can even make sense out of an outward or inward normal.

We now define the surface integral of a vector field F on the surface S by defining it as the integral of the function $F \cdot n = F \cdot \frac{\nabla(g)}{|\nabla(g)|}$. Our formula becomes

$$\int\int_S F \cdot n |\nabla(g)| \, |\omega_g| = \int\int_D F \cdot \nabla(g) \, |\omega_g|.$$

For a parametric surface, the formula becomes:

$$\int\int_D F(r(u,v)) \cdot r_u \times r_v \, dA.$$

Note that the above formula presumes a certain direction of the unit normal as determined by $\nabla(g)$ or $r_u \times r_v$. We multiply by -1 if we wish to change the direction.

4 Stokes' Theorem.

Green's theorem equates the line integral around a plane curve with the integral on the enclosed region, which is a flat surface. Stokes' theorem gives a similar relation for a space curve bounding a surface. Indeed, the Green's theorem is but a special case of Stokes' theorem.

The theorem states:

$$\int_C F \cdot dr = \int\int_S \text{ curl } F \cdot n \, dS$$

where S is an oriented surface enclosed by the curve C and we use the induced orientation on C.

This is best imagined as follows. Assume that you are walking on the curve C such that the chosen normal points overhead. Then the curve C is positively oriented, if the surface lies on your left hand side!

Let us further note that if $g(x,u,z) = 0$ is the equation of the surface, then $\text{ curl } F \cdot n \, dS = \text{ curl } F \cdot \nabla g \, \omega_g$ as explained above.

The proof of Stokes' theorem is usually deduced by cutting up the surface into small enough pieces so that each piece can be arranged to be parametrized by (x,y) or (y,z) or (z,x). Then the proof reduces to a plane integral and works as in the Green's theorem.

The Green's theorem itself is also established by cutting up the plane region into pieces of type I or II (i.e. where the double integration is done by vertical or horizontal sections between parallel lines.)

One of the most useful consequences of the Stokes' Theorem is that it says that if two surfaces share a common oriented boundary curve, then the integral of a vector field of the form $\text{ curl } F$ is the same on both of them.

Then, integrals of such vector fields can be conveniently evaluated by changing the surface to something more convenient.

Example. Consider the surface $S : z = x^2 + y^2 - 1$ below the (x,y)-plane. Let $G =< 1, 1, 1 >$ and consider $\int\int_S G \cdot n \, dS$

It is easily seen that $G = \text{ curl } < z, x, y >$ the surface integral can be easily evaluated as a line integral along the boundary curve $C :< \cos(t), \sin(t), 0 >$ as t goes from 0 to 2π and evaluates to π.

Now the surface can be made a lot more complicated, say $z(1 + x^4 + x^2y^2 + y^4) = x^2 + y^2 - 1$ and the corresponding surface integral is quite messy! But we know its value already.

One of the trivial but useful consequences is that the integral of any field of the form $\text{ curl } F$ on a closed orientable surface is zero. Thus, the integral of a constant vector field on such a surface is seen to be zero!

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