MA213
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Notes on Using vectors for Geometry

We discuss some sample calculations from old Geometry using vectors. We assume that A, B, C denote position vectors for three vertices of a proper triangle. These may be assumed to be in a plane, but this works as well in higher dimensional space.

In the usual convention, we let

$$a = |\overrightarrow{BC}| = |C - B|, b = |\overrightarrow{CA}| = |A - C|, c = |\overrightarrow{AB}| = |B - A|.$$

- **Medians of a triangle.**

  The midpoint A^* of the segment BC is easily seen to be $\dfrac{B+C}{2}$.

  Any point on the line AA^* is then given by $(1-t)A + t(\dfrac{B+C}{2})$ for a real t. Consider $t = \frac{2}{3}$. then we get the point

  $$(1 - \frac{2}{3})A + \frac{2}{3}(\frac{B+C}{2}) = \frac{A}{3} + \frac{B}{3} + \frac{C}{3}.$$

  By symmetry, we see that this point is on all three medians and this proves that the medians are concurrent and their intersection is a trisection point on each of them.

- **A point inside the triangle** We next show that any point inside the $\triangle ABC$ is given as $pA + qB + rC$ for some non negative p, q, r such that $p + q + r = 1$.

  We already know that we get points on the sides AB, BC, CA by choices with $r = 0$, $p = 0$ and $q = 0$ respectively.

  Note that
  $$\tilde{A} = \frac{q}{1-p}B + \frac{r}{1-p}C$$
  is a point on the line BC since we have:

  $$\frac{q}{1-p} + \frac{r}{1-p} = \frac{q+r}{1-p} = \frac{1-p}{1-p} = 1.$$

  It follows that $pA + qB + rC = pA + (1-p)\tilde{A}$ is a point on $A\tilde{A}$ and hence is in the inside of the triangle.

  Converse is proved by reversing these steps.

- **Isosceles triangle** Suppose that we have $b = c$ so the triangle is isosceles with vertex at A. Then we shall prove that $\angle B = \angle C$.

  1. First note that $b = c$ means $|AC|^2 = |AB|^2$ and hence:

     $$(C - A) \cdot (C - A) = (B - A) \cdot (B - A) = (A - B) \cdot (A - B) \tag{1}$$

  2. Note that $(C - B) \cdot (A - B) = ac\cos(\angle B)$ and $(A - C) \cdot (B - C) = ab\cos(\angle C)$. We shall show that the left hand sides of these equations are equal, so the right hand sides are equal and thus the angles coincide, since $ab = ac$.

3. Moreover

$$(C - B) \cdot (A - B) = (C - A) \cdot (A - B) + (A - B) \cdot (A - B) \tag{2}$$

Using equation (1), we see that this equals

$$(C - A) \cdot (A - B) + (C - A) \cdot (C - A) = (C - A) \cdot (C - B) = (A - C) \cdot (B - C) \tag{3}$$

This last equation proves the desired result.

The above proof may seem complicated, but it is pure Algebra and does not depend on geometric intuition. Moreover, it then works in any vector space with inner product.

The converse can be similarly established.

- **Law of Sines** We shall prove the famous identity

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}.$$

Note that we are using the cross product here and we know that $|v \times w| = |v||w||\sin(\theta)|$ where θ is the angle of rotation from v to w.

First note that $|(B - A) \times (C - A)| = cb|\sin(\angle A)| = cb\sin(\angle A)$ since sine of angles of a triangle is always positive. Moreover, this is evidently equal to $abc\frac{\sin(\angle A)}{a}$. So, it is enough to prove

$$|(B - A) \times (C - A)| = |(C - B) \times (A - B)| = |(A - C) \times (B - C)|.$$

But we note that

$$(B - A) \times (C - A) = B \times C - A \times C - B \times A + A \times A = B \times C + C \times A + A \times B + 0$$

using the properties of cross products.

The last expression is completely symmetric in A, B, C and it is clear that all the three vectors under consideration will be equal, so their magnitudes will be also equal.

We already know that the magnitude is the area of the triangle $\triangle ABC$. Hence, we deduce that the common value of the three ratios is also equal to $\frac{\text{Area}(\triangle ABC)}{abc}$

Try to do as many theorems as possible from the old geometry books, using algebra of vectors!