General Information. **Subject to amendment**

The exam 2 will have 6 questions. You should study using WHS problems, quizzes, class notes as well as on line notes and your own work.

Generally, you should simplify answers and do correct algebraic modifications. You are expected to know the basic values from trigonometry and the standard identities. You must do simplifications when possible, but approximate decimal answers are not recommended. Thus, when I have provided decimal answers below, the only reason is not to give away the intermediate steps and thus force you to carry out the steps yourself and use the final answer only as a confirmation.

You are permitted to bring a two sided sheet of notes of your own, as before.

1. Given position vector for a particle in space $P(t) =< t^2 + 5t, -t^2 + 3t, 2t + 3 >$, calculate the following quantities. Generally, it is OK to leave radicals unevaluated, but it is good to simplify expressions so that later calculations are manageable.

   (a) Velocity vector $v$ and speed $|v|$. **Answer:** $< 2\,t + 5, -2\,t + 3, 2 >$, $\sqrt{38 + 8\,t^2 + 8\,t}$.

   (b) Find acceleration $a$. **Answer:** $< 2, -2, 0 >$.

   (c) Find $B$. **Answer:** $H = v \times a =< 4, 4, -16 >$. So $B = \frac{1}{12\sqrt{2}} < 4, 4, -16 >$ .

   (d) Find unit normal $N$. **Answer:** $H \times v =< 56 - 32\,t, -32\,t - 88, -16\,t - 8 >$. To get $N$, divide by $24\,\sqrt{19 + 4\,t^2 + 4\,t}$.

   (e) Calculate $T, B, N$ at $t = 0$. **Answer:** $v(0) =< 5, 3, 2 >$, so $T =< 5, 3, 2 > /\sqrt{38}$. Similarly, $N(0) =< 56, -88, -8 > /24\sqrt{19} =< 7, -11, -1 > /3\sqrt{19}$.

   (f) Calculate the curvature at $t = 0$. **Answer:** $|v(t)| = \sigma(t) = \sqrt{38 + 8\,t^2 + 8\,t}$. So $<$ $|H(0)|/\sigma^3(0) = \frac{12\sqrt{2}}{38\sqrt{38}}$.

   (g) Is this a plane curve? Explain. **Answer:** Since $B$ is a constant vector, the motion is plain. Explicitly, note that $H(t) \cdot (P(t) - P(0)) = 0$. This says that if we write $< x, y, z >= P(t)$, then $H(t) \cdot (P(t) - P(0)) =< 4, 4, -16 > \cdot < x, y, z - 3 >= 0$. This is the plane!

   (h) Find the tangential and normal components of $a$ at $t = 0$, **Answer:** $a_T = \frac{a \cdot v}{v \cdot v} v = \frac{2}{19} <$ $5, 3, 2 >$. The normal component is best found as $a - a_T =< 2, -2, 0 > -\frac{2}{19} < 5, 3, 2 >$.

2. Find where the function $\frac{x^3+y^3+xy}{x(x+y)(x^4+y^4)}$ is continuous. **Answer:** Since it is a rational function, it is continuous except when the denominator is zero, i.e. outside the $y$ axis and the line $x + y = 0$.

3. Find where $\log(\frac{1-xy}{1+x^2y^2})$ is continuous. **Answer:** We need $\frac{1-xy}{1+x^2y^2}$ to be positive. Since the denominator is always positive, it is enough to check when $1 - xy > 0$. This is the region outside of the two jaws of the hyperbola $xy = 1$.

4. What are the level curves of the function $z = x^2 + y^2 + 2x + 6y$. Deduce the range of the function.**Answer:** The curves $c = x^2 + y^2 + 2x + 6y$ are rewritten as $(x + 1)^2 + (y + 3)^2 = c + 10$. These are circles centered at $(-1, -3)$ as long as $c \geq -10$. So, the range is $[-10, \infty)$.

5. Find the limit as $(x, y) \to (0, 0)$ if it exists. If it does not, then explain why. $f(x, y) = \log(\frac{x^2+y^2+4}{2x+3y+5})$. **Answer:** The function is continuous near $(0, 0)$ so evaluate: $\log(4/5)$.

6. Find the limit as $(x, y) \to (0, 0)$ if it exists. If it does not, then explain why. $f(x, y) = \frac{y^4}{x^4+y^4}$. **Answer:** Try approach along lines $y = mx$. We get $\lim_{x \to 0}\left(\frac{m^4}{1 + m^4}\right) = \frac{m^4}{1 + m^4}$. Since this varies with $m$, the limit DNE! We could also try polar coordinates with the same result.

7. Find the limit as $(x, y) \to (0, 0)$ if it exists. If it does not, then explain why. $f(x, y) = \frac{xy^4}{x^2+y^2}$.
**Answer:** Try approach along lines $y = mx$. We get $\lim_{x\to 0}\left(x^3 \frac{m^4}{1+m^2}\right) = 0$. This may suggest existence of limit 0. To confirm, try polar coordinates. $\lim_{r\to 0}\left(r^5 \frac{\cos(\theta)\sin^4(\theta)}{r^2}\right)$. Canceling $r^2$, we get $\lim_{r\to 0} r^3 \left(\cos(\theta)\sin^4(\theta)\right)$ and this is seen to be 0, since the $r^3$ goes to 0, while the second factor is bounded.

8. Find the limit as $(x, y) \to (0, 0)$ if it exists. If it does not, then explain why. $f(x, y) = \frac{xy^2}{x^2+y^4}$.
**Answer:** Try approach along lines $y = mx$ and see that the limit appears to be zero as before. So, the polar substitution is tried. This gives $\lim_{r\to 0}\left(\frac{r\cos(\theta)\sin^2(\theta)}{\cos^2(\theta)+r^2\sin^4(\theta)}\right)$. If we let $\theta$ go near $\pi/2$, then the denominator can be made zero, so there might be a problem. We use the curve $r = m\cos(\theta)$ and get the simplified fraction: $\frac{m^2\sin^2(\theta)}{1+m^2\sin^4(\theta)}$, and this clearly has variable limit as $r$ goes to 0 and $\theta$ goes to $\pi/2$. So DNE!

9. Let

$$f(x, y) = x^3 - 3y^3 + xy + 5x, x = g(u, v) = 3uv + u + v, y = h(u, v) = u^2 - 2v^2, P(x, y, z) = x^3 + y^2 + xyz.$$

Answer the following questions.

(a) Find $\nabla(f), \nabla(g), \nabla(h)$. **Answer:** $\nabla(f) =< 3\,x^2+y+5, -9\,y^2+x >, \nabla(g) =< 3\,v+1, 3\,u+1 >, \nabla(h) =< 2\,u, -4\,v >$.

(b) Find $\frac{\partial f}{\partial u}$. **Answer:** $(3\,x^2 + y + 5)\,(3\,v + 1) + 2\,(-9\,y^2 + x)\,u$.

(c) Find the value of $\frac{\partial f}{\partial u}$ when $u = 2, v = -1$. **Answer:** First calculate $x = -5, y = 2$. Then use the above answer: $-328$.

(d) Calculate the Laplacian of each of these functions. **Answer:** $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x - 18y$. Similarly, $\nabla^2 g = 0$ and $\nabla^2 h = -2$.

(e) What is $D_u(P)(1, -2, -3)$ when $u =< 1, -2, 1 >$? Which directions give the smallest and the largest directional derivative? Which direction gives zero derivative? **Answer:** $D_u(P)(1, -2, -3) = 21/\sqrt{6}$. Largest derivative is along $< 9, -7, -2 >$ with value $\sqrt{9^2 + 7^2 + 2^2}$. the least is in the opposite direction and its value is the negative of the largest value. derivative is zero for all directions perpendicular to $< 9, -7, -2 >$.

10. If $x^2 + y^3 + z^5 + 6xyz = 9$, then find $\frac{\partial z}{\partial x}$. Also evaluate it when $x = y = z = 1$. **Answer:** We have $(2\,x + 6\,yz)D(x) + (3\,y^2 + 6\,xz)D(y) + (5\,z^4 + 6\,xy)D(z) = 0$. Then $\frac{\partial z}{\partial x} = -\frac{2\,x+6\,yz}{5\,z^4+6\,xy}$ Evaluation, gives $-\frac{8}{11}$.

11. Volume of a pyramid is 1/3 times the area of its base times its height. If $f(x, y)$ is the volume of a pyramid whose base is a square of side $x$ and has height $y$, use differentials to find a linear approximation function for the volume when $x = 35$ and $y = 15$. Use this to estimate the volume of a pyramid whose base has side 36 and height 14. **Answer:** $f(x, y) = (1/3)x^2y$. $L(x, y) = (1/3)(35)^2(15) + (1/3)(2(35))(15)(x - 35) + (1/3)(35^2)(y - 15)$. Estimate is 6066.67.

12. What is the equation of the tangent plane to $x^2 + y^2 - z^2 + xy + x + z + 8 = 0$. Find the tangent plane to it at $(x, y, z) = (1, -2, -3)$. Using the tangent plane approximation, **estimate** $c$ if $(1.2, c, -2.8)$ is on the surface. **Answer:** Tangent plane is $(x-1) - 3(y+2) + 7(z+3) = 0$. Substituting in this equation, we get $c = -(4.4)/3$.

13. Find all the critical points of these functions.

$$f(x,y) = x^2+4xy+y^2-2x-3y, g(x,y) = x^2-y^3+3\,y+x, h(x,y,z) = x^2-3\,y^2+z^2+xy+yz-7\,z.$$

**Answer:** For $f$, we solve $\{2\,x+4\,y-2=0, 4\,x+2\,y-3=0\}$ and the unique solution is $(4/6, 1/6)$. For $g$, we solve $\{2x+1=0, -3y^2+3=0\}$. We get two critical points $(-1/2, 1), (-1/2, -1)$. For $h$, we solve $\{2\,x+y=0, -6\,y+x+z=0, 2\,z+y-7=0\}$ and the solution comes out $(-1/4, 2/4, 13/4)$.

14. Testing for max/min. Consider the function $f(x,y) = 3x^2 + axy + 4y^2 + x^2 * y^2$ where $a$ is a constant. Show that $f(x,y)$ has a critical point at $(0,0)$ and determine its nature for various values of $a$. Note that four answers are possible: 1. Local max. 2. Local min. 3. Saddle point. 4. Test is inconclusive.

**Answer:** First calculate the first and the second derivatives of $f$. We have:

$$f_x = 6x + ay + 2xy^2, \quad f_y = ax + 8y + 2x^2y.$$

This gives values 0 at $(0,0)$, hence we have a c.p.

Now

$$f_{xx} = 6 + 2y^2, \quad f_{xy} = f_{yx} = a + 4xy, \quad f_{yy} = 8 + 2x^2.$$

Their values at $(0,0)$ give:

$$f_{xx} = 6, \quad f_{xy} = f_{yx} = a, \quad f_{yy} = 8.$$

Now our test value $D = (6)(8) - a^2 = 48 - a^2$.

Now we answer everything:

- If $|a| > \sqrt{48}$ then $D < 0$, so we have a saddle point.
- If $|a| = \sqrt{48}$, then $D = 0$ and the test is inconclusive.
- If $-\sqrt{48} < a < \sqrt{48}$ then we have local max. or local min. Since $f_{xx} > 0$, we have a local min. If it were negative, then we would have a local max.

15. Consider similar problems by taking parameters like $a, b$ etc. in different places. For example, analyze $2x^2 + 3axy + (4-a)y^2$.

Also note that terms of degree bigger than 2 can be inserted at will, when you are analyzing the function at $(0,0)$.

**Answer:** We calculate $D$ for the shown function. $D = (4)(2(4-a)) - 9a^2$. The analysis can be carried out by solving the inequality $D < 0$ and $D > 0$.