General Information.

The examination will have 6 questions. You should study using WHS problems, quizzes, class notes as well as on line notes and your own work.

Generally, you should simplify answers and do correct algebraic modifications. Often, it is OK to leave radicals and other calculator functions like arccos unevaluated. You are expected to know the basic values from trigonometry and the standard identities. You must do simplifications when possible, but approximate decimal answers are not recommended. Thus, when I have provided decimal answers below, the only reason is not to give away the intermediate steps and thus force you to carry out the steps yourself and use the final answer only as a confirmation.

You are permitted to bring your own two sided sheet of notes, as before. No other material may be used during the exam.

1. Find all the critical points of the following functions and use the second derivative test to determine if it is an extremum point and if so, of what type. If the test is inconclusive, explain why.

   (a) $f(x,y) = x^3 + y^3 - 3\,xy + 4$. **Answer:** Point $(0,0)$ $D = -9$ so saddle pt. Point $(1,1)$ $D = 27$ so extremum. $f_{xx} = 6$ is positive, so local min.

   (b) $f(x,y) = x^2 + 2\,xy + y^2 + x^3$. **Answer:** Critical point $(0,0)$. $D = 0$, so inconclusive test. Careful analysis near the point shows a saddle point!

   (c) $f(x,y) = \mathrm{e}^{4\,y-x^2+y^2}$. **Answer:** Point $(0,-2)$. $D = -4\,(\mathrm{e}^{-4})^2 < 0$. So saddle point.

2. Use Lagrange Multipliers to solve the following problems.

   (a) Find the maximum volume of a rectangular box with $(0,0,0)$ and (x,y,z) as opposite corners where x,y,z satisfy $5x + 4y + 5z = 60$. **Answer:** A real box would have positive x,y,z. The Lagrange function is $F(x,y,z) = xyz - \lambda(5x+4y+5z-60)$. This gives only one critical point $(4,5,4)$ with volume 80. Note that the function does not have an absolute maximum or minimum, if we allow arbitrary x,y,z.

   (b) Find the minimum surface area of a box without a top of volume 500 cubic inches. **Answer:** The Lagrange function is $F(x,y,z) = 2xz + 2yz + xy - \lambda(xyz - 500)$. Naturally, x,y,z are positive. The only point is $(10,10,5)$ with area 300.

   (c) Find the distance of the origin from the surface $xyz = 1000$. **Answer:** The Lagrange function is $F(x,y,z) = x^2 + y^2 + z^2 - \lambda(xyz - 1000)$. Note that we use the square of the distance function since it would lead to the same minimum point. Also, we may clearly assume our coordinates to be all positive to get the minimum distance.
   Then the only point is $(10,10,10)$ which gives the distance of $\sqrt{300}$.

3. Sketch the area of integration of the given integral and reverse the order of integration. Evaluate the integral.

   (a) $\displaystyle\int_{x=0}^{1}\int_{y=1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy\,dx$. Also, evaluate the mass if the density function is $1+x$.
   Answer: The region is the area between the parabola $x = 2y - y^2$ and y-axis.
   The reversed integral is $\displaystyle\int_{y=0}^{2}\int_{x=0}^{2y-y^2} dx\,dy = 4/3$

The mass evaluates to $\int_{y=0}^{2} \int_{x=0}^{2y-y^2} (1+x)\, dx\, dy = 28/15$.

(b) $\int_{y=0}^{5} \int_{x=-y/5}^{y/5} dx\, dy$..

Also, evaluate the mass and the center of mass if the density function is y^2.

Answer: The region is the triangle with vertices $(-5,5),(5,5),(0,0)$.

The reversed integral has two parts. $\int_{x=-1}^{0} \int_{y=-5x}^{5} dy\, dx + \int_{x=0}^{1} \int_{y=5x}^{5} dy\, dx = 5$.

The mass evaluates to $m = \int_{y=0}^{5} \int_{x=-y/5}^{y/5} y^2\, dx\, dy = 125/2$.

For center of mass, calculate $m\overline{x} = \int_{y=0}^{5} \int_{x=-y/5}^{y/5} xy^2\, dx\, dy = 0$

and $m\overline{y} = \int_{y=0}^{5} \int_{x=-y/5}^{y/5} y^3\, dx\, dy = 250$.

Hence, the center of mass is $(\frac{0}{125/2}, \frac{250}{125/2}) = (0,4)$.

4. Calculate the indicated integrals. Use polar/cylindrical/spherical coordinates as appropriate, or other convenient changes of coordinates. Do practice writing limits in the original coordinates.

(a) $\iiint_R \sqrt{x^2+y^2}\, dv$ where R is the region inside the sphere $x^2+y^2+z^2=9$ and the first octant.

Answer: It is evident that the spherical coordinates will be useful. The region of integration in the (ρ,ϕ,θ) space is given by $0 \le \rho \le 3$, $0 \le \phi \le \pi/2$ and $0 \le \theta \le \pi/2$.

The integral becomes $\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{3} \rho \sin(\phi) \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = 81\pi^2/32$.

(b) $\iiint_R x dv$ where R is the region bounded by $x^2+y^2=1$, $z=0$ and $z^2=4(x^2+y^2)$.

Answer: It is evident that cylindrical coordinates will be useful, since the xy projection is the unit circle. The region becomes $0 \le r \le 1$, $0 \le \theta \le 2\pi$ and $0 \le z \le \sqrt{4(x^2+y^2)} = 2r$.

The integral is $\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{2r} r\cos(\theta) r\, dz\, dr\, d\theta = 0$.

A little thought could have predicted this since we could carry out the integration with respect to θ first.

Note: Consider changing the x to y, z and 1 respectively. These calculations will let us compute the mass and center of mass, assuming a constant density function 1.

Verify:

$$m = \iiint_R x dv = 4\pi/3, m\overline{x} = \iiint_R x dv = 0, m\overline{y} = \iiint_R y dv = 0, m\overline{z} = \iiint_R z dv = \pi.$$

Thus, the center of mass is $(0,0,\frac{\pi}{4\pi/3}) = (0,0,\frac{3}{4})$. Of these three numbers, the first two zeros could be guessed from symmetry.

(c) $\displaystyle\iiint_R x dv$ where R is the region bounded by $x^2 + y^2 = 1$, $z^2 = 4(x^2 + y^2)$ and lying in the first octant.

Answer: The problem is practically the same as above, but the condition of the first octant changes the limits. The integral in cylindrical coordinates becomes

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{z=0}^{2r} r\cos(\theta) r \, dz \, dr \, d\theta = 1/2.$$

(d) $\iint_R (x+y) dx\, dy$ where R is the parallelogram with vertices $(0,0), (4,5), (2,3), (6,8)$. Do this by transforming the region R to the unit square S with corners $(0,0), (1,0), (0,1), (1,1)$.

Answer: We make a substitution $(x, y) = T(r, s) = (ar + bs, cr + ds)$. We have equations:

- To make $T(1,0) = (4,5)$, $a = 4, c = 5$.
- To make $(0,1) = (2,3)$ $b = 2, d = 3$.

The Jacobian of T is $\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = 2$. Thus, our integral is: $\displaystyle\int_0^1 \int_0^1 (4r + 2s + 5r + 3s)|2| dr\, ds =$ 14.

5. Additional formulas on surface area would be posted in a separate file soon.