Quiz 20 Fundamental Theorem.

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Fundamental Theorem.

Suppose that we have a linear transformation $L : V \to W$ where V has finite dimension $n > 0$. Let $H_1 = Ker(L)$ be the subspace of V and $H_2 = Image(L)$ the subspace of W. Then

$$\dim(V) = \dim(H_1) + \dim(H_2).$$

Verify the theorem for the following example.

1. Let $V = Span\{\sin(x), \sin(2x), \cos(x), \cos(2x)\}$ and $W = V$. Consider $L : V \to W$ defined by $L(f(x)) = f''(x) + 4f(x)$. Determine $Ker(L)$ and $Image(L)$.

   Answer: $Ker(L) = Span\{\sin(2x), \cos(2x)\}$ and $Image(L) = Span\{\sin(x), \cos(x)\}$.

2. Determine the dimensions of $Ker(L)$ and $Image(L)$.

   Answer: Since each has 2 generators, the dimension is at most 2. But they are linear independent since the ratios of the two generators are $\tan(2x)$ and $\tan(x)$ are not constants.

3. Explain why $\dim(V) = 4$. You may not have a complete proof during the quiz, but it should at least be done at home.

   Answer: Long way: Make a linear combinations of the four generators and get four equations by plugging in special values for x. Deduce that the resulting linear equations have only trivial solutions.

   shortcut: We can use the fundamental theorem and deduce that the dimension of V is $2+2=4$.

4. For meditation: Think how one may be able to prove

$$\sin(x), \cos(x), \sin(2x), \cos(2x), \cdots, \sin(nx), \cos(nx)$$

are independent for all integers $n \geq 1$. This will need a bit deeper meditation!