1. Given any square matrix $A = (a_{ij})$, we define its **determinant** variously denoted as $\det(A)$ or $|A|$ or $||A||$. The definition needs some auxiliary terms. First for any (i, j) such that a_{ij} is defined, we define the **index of** $(\mathbf{i}, \mathbf{j})$ to be $\text{ind}(i, j) = (-1)^{(i+j)}$. Further we define the **minor of** $(\mathbf{i}, \mathbf{j})$ to be $\text{minor}(A, i, j)$ which is the subdeterminant of A obtained by throwing away the i-th row and the j-th column. We also define the **cofactor of** $(\mathbf{i}, \mathbf{j})$ to be $\text{cofactor}(A, i, j) = \text{ind}(i, j)\text{minor}(A, i, j)$.

   Then the determinant $|A|$ can be computed as $\sum a_{ij}\text{cofactor}(A, i, j)$ where the sum is taken over all entries a_{ij} coming from any chosen row or column.

2. The value of the determinant gets multiplied by k if all entries in a single row or column are multiplied by k.

3. The value of the determinant gets multiplied by -1, if a single exchange of two rows or two columns is carried out. For more complicated permutations, we multiply by the sign of the permutation.

4. The value of the determinant is unchanged if we add a multiple of one row to another. Similar result holds for columns.

5. The value of a determinant is 0 if some row or column consists of zero entries only! (This follows from the definition.)

6. The value of a lower triangular determinant is equal to the product of its diagonal entries. Ditto for upper triangular. In general, this is how determinants are computed: reduce the determinant to upper or lower triangular form and then evaluate the product of the diagonal entries. If permutations are used along the way, then suitable sign is attached to the answer. Sometimes, an expansion along a suitable row/column is also used to reduce the work.

7. There is a more general expansion, the so-called Laplace expansion, which works with several rows (or columns) at once, instead of the single row (or column) as in the definition.

8. The adjoint of a matrix A is a matrix denoted by A^{adj} whose (i, j)-th entry is equal to $\text{cofactor}(A, j, i)$. **Do notice the switch in the order!** The adjoint satisfies the identity

   $$AA^{\text{adj}} = A^{\text{adj}}A = |A|I.$$

   This lets us write the inverse of A as $A^{\text{adj}}/|A|$. Of course it exists iff $|A| \neq 0$.

9. For a general matrix M **its rank** $\text{rank}(M)$ is defined to be the largest number r such that M has a nonzero subdeterminant of size r. Thus a square $n \times n$ matrix is invertible iff its rank is n. Rank of a matrix is obviously less than or equal to its rowdim as well as coldim.

10. In general, **the equations** $\mathbf{AX} = \mathbf{B}$ **are solvable** iff $\text{rank}(A) = \text{rank}(A|B)$. Here $A|B$ stands for the augmented matrix. Obviously, if $\text{rank}(A) = \text{rowdim}(A)$ then $AX = B$ is solvable for all B. The converse is true too!