MA322
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For memorizing.

1 Solution process.

1. A pivot of a row of a matrix is the first non zero entry in that row. The pc (pivot column) is the column number of the pivot.

   If a row has all zeros, then it has no pivot and the pc number is defined to be ∞.

2. A PC list of a matrix is a sequence of column numbers of pivots in successive rows of the matrix. Note that some entries of a pc list may be ∞.

3. A pc list is said to be strict if its successive finite entries are strictly increasing and no infinite entry appears before a finite entry.

4. A matrix is said to be in REF (row echelon form) if its pc list is strict.

   It is said to be in RREF (reduced row echelon form), if it is in REF, all pivot entries equal 1 and if a column has a pivot in it, then all other entries in the column are zero.

5. For any matrix A, defined $rownum(A)$ = number of rows of A and $colnum(A)$ = number of columns of A.

   The rank of a matrix A is defined to be the number of pivots in M where M is an REF of A. In this form, it is not clear if it is well defined, until we produce alternate equivalent definitions later.

6. It is evident that every row has at most one pivot and so the rank of a matrix is at most $rownum(A)$. Similarly, a column also has at most one pivot, so rank is at most $colnum(A)$.

7. $\Re^m$ is defined as the set of all columns of height m with real entries.

8. If A is a matrix with columns $C_1, C_2, \cdots, C_n$ in $\Re^m$ and B is some column also in $\Re^m$, then the augmented matrix $(A|B)$ is said to define a system of linear equations $x_1C_1 + \cdots + x_nC_n = B$ in variables $x_1, \cdots, x_n$.

   A linear system may have parameter entries. These are entries involving variables which are treated as unspecified real numbers.

   The same system may be compactly written as $AX = B$ where X denotes a column of the variables $x_1, \cdots, x_n$ in order.

   $(A|B)$ is said to be a homogeneous linear system if B is the zero vector.

9. A linear system $(A|B)$ is said to be consistent if there is a vector v such that the equations $Av = B$ hold.

   Evidently $(A|0)$ is always consistent since we can take $v = 0$.

10. A solution of $(A|B)$ is any vector v such that $Av = B$. A general (or complete) solution is an expression $X = X_p + s_1X_1 + \cdots + s_lX_l$ where:

    - X_p is some particular solution.
    - $X_1, \cdots, X_l$ are solutions of the associated homogeneous system $(A|0)$, i.e. $AX_i = 0$ for all $i = 1, \cdots, l$.
      The part $s_1X_1 + \cdots + s_lX_l$ is said to be the homogeneous part of the solution.
    - The columns $X_1, \cdots, X_l$ are linearly independent (defined later). The solution process using REF (or RREF) automatically makes independence.
    - **Any solution** v such that $Av = B$ can be written as $v = X_p + a_1X_1 + \cdots + a_lX_l$ where $a_1, \cdots, a_l$ are scalars. Moreover, the scalars $a_1, \cdots, a_l$ are uniquely determined once $X_p, X_1, \cdots, X_l$ are chosen.

11. For a matrix A, we define $Nul(A) = \{X|AX = 0\}$, i.e. the set of all solution of the homogeneous system $(A|0)$.

    We also define $Col(A) = \{AX|X\}$

    If $rownum(A) = m$ and $colnum(A) = n$, then note that $Nul(A)$ is contained in $\Re^n$ while $Col(A)$ is contained in $\Re^m$.

12. A set of columns $C_1, \cdots, C_n$ are said to be linearly dependent if there are scalars $a_1, \cdots, a_n$ such that $\sigma_{i=1}^n a_i C_i = 0$ and at least one of $a_1, \cdots, a_n$ is non zero.

    We can equivalently say that $C_1, \cdots, C_n$ are said to be linearly dependent if the matrix A with these columns has a nonzero vector in its null space, i.e. $Nul(A)$ is not zero

13. A set of columns $C_1, \cdots, C_n$ are said to be linearly independent if for any scalars $a_1, \cdots, a_n$, if $\sigma_{i=1}^n a_i C_i = 0$, then each a_i is equal to zero.

    Equivalently, the null space $Nul(A)$ of the matrix formed by the columns is equal to zero!

2 Vector spaces

The spaces $\Re^n$ described above are the basic examples of vector spaces. Now we extend the notions to general vector spaces with more useful formulas and definitions.

1. A vector space is a set with a well defined operation of addition and scalar multiplication where scalars are usually real numbers.

   In case of $\Re^n$, this is simply the addition of (column) matrices and multiplication of scalars.

2. If V is a vector space, then a subset W of V is said to be a subspace if W is closed under addition and scalar multiplication borrowed from V.

   This means, for any given w_1, w_2 in W and any scalar c, we know that $w_1 + w_2 \in W$ and $cw_1 \in W$.

3. we already know two important subspaces of $\Re^n$.

   If $A = A_{m \times n}$ is a matrix, the $Col(A)$ is a subspace of $\Re^m$ and $Nul(A)$ is a subspace of $\Re^n$.

   Verify this and be prepared to compute both, for any given A.

4. Suppose V is a vector space and $v_1, \cdots, v_s$ is any set of vectors in V.

   The set of all linear combinations $\sum_i a_i v_i$ with scalars $a_1, \cdots a_s$ is called $Span\{v_1, \cdots, v_s\}$.

   Briefly, this is denoted as $\{\sum_i a_i v_i \mid a_i \text{ are scalars } \}$.

   This should be verified to be a subspace of V.

   The definition easily extends to an infinite set S: $Span(S) = \{\sum_i^m a_i v_i \mid \text{ such that each } v_i \in S, m \geq 0 \text{ and } a_i \text{ are scalars. } \}$

5. A set $\{v_1, \cdots, v_s\}$ of vectors in V is said to be a spanning set for V if every $v \in V$ satisfies $v = \sum_i a_i v_i$ for some scalars $a_1, \cdots, a_s$. This clearly means every $v \in V$ is also in $Span\{v_1, \cdots, v_s\}$ or equivalently $V = Span\{v_1, \cdots, v_s\}$.

6. We define independence and dependence of vectors similar to the definition of column vectors. Just replace columns C_i by vectors v_i.

   If V is a vector space and $v_1, \cdots, v_n$ are in V, we say that $\{v_1, \cdots, v_n\}$ is a basis of V if

   - $Span\{v_1, \cdots, v_n\}$ is a spanning set for V and
   - $\{v_1, \cdots, v_n\}$ is an independent set of vectors. For convenience we write a sequence of these vector and call it a basis of V.

     Thus we say B is a basis of V, where $B = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.

     Note that in this notation, a change of order gives a "new" basis!

7. Now we list some important results and calculations.

   (a) The number of vectors in a basis is said to be the dimension of V. It will be established that this number is uniquely defined by the chosen vector space.

   (b) If you have a matrix A with columns $C_1, \cdots, C_n$ then it is possible to identify the pc list without row reductions.

   Define $d_i = \dim(Span\{C_1, \cdots, C_i\})$ for $i = 1, \cdots, n$. It is not hard to see that $d_1 \leq d_2 \leq \cdots \leq d_n$.

   The pivot columns are given by those values of i for which $d_{i-1} < d_i$, i.e. places where dimension jumps.

   The columns corresponding to the free variables are the ones where the dimension stutters (does not jump).

   (c) Challenge: Can you prove that $d_i - d_{i-1} \leq 1$ for all i?

(d) The rank of a matrix A can now be defined as $\dim(Col(A))$. A basis for the $Col(A)$ can always be chosen as the columns in A. which become pivot columns in its REF. **It is crucial not to use the columns in REF as a basis. They may not even be in the column space of A.**

(e) The number $colnum(A) - rank(A)$ gives the dimension of the null space $Nul(A)$. It is equal to the number of non pivot columns (in REF) and equals the number l described in the general solution of $(A|B)$.

3 Linear Transformations.

1. Let V and W be vector spaces (over the same field of scalars). A function $L : V \to W$ is said to be a linear transformation if the following conditions hold for all $v_1, v_2 \in V$ and all scalars a.
   - $L(v_1 + v_2) = L(v_1) + L(v_2)$ and
   - $aL(v_1) = L(av_1)$
2. We define a subspace of V called $Ker(L)$ by:

   $Ker(L) = \{v \in V \mid$ such that $L(v) = 0$.

   You should prove that this is indeed a subspace of V.

   We also define a subspace of W by:

   $Image(L) = \{w \in W \mid w = L(v)$ for some $v \in V\}$.

   You should prove that this is indeed a subspace of W.
3. The title "fundamental theorem of linear algebra" may be given to the statement:

   $\dim(V) = \dim(Ker(L)) + \dim(Image(L))$.

to be extended.