DEPARTMENT OF MATHEMATICS

Ma 322 Solved First Exam February 17, 2017
With supplements

1. **Points: 4+7+4**

You are given an augmented matrix of a linear system of equations. Here t is a parameter:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -4 & 1 & -4-t & 0 \\ -3 & 2 & -3+t & 0 \end{array}\right]$$

answer the following questions

(a) Carefully define what is meant by a system of linear equations to be consistent.

A Linear system of equations may be represented as $AX = B$ where A is an $m \times n$ matrix of scalars, X is a column of n variables and B is a column of m scalars.
It is said to be consistent if there is a vector $v \in \Re^n$ such that $Av = B$.

For a scalar field K other than $\Re$ the vector is assumed to be in K^n.
(b) Explain why the above linear system of equations is consistent for all values of t. Be sure to explain how your conclusion is consistent with you definition.

Our current system is of the form $AX = 0$. If we take v to be the zero vector in $\Re^n$, then $Av = 0$ is true, regardless of the entries of A. Hence it is consistent for all values of t.

(c) Find all values of t for which the above linear system has **infinitely many solutions.** Be sure to justify your answer.

Since a system $AX = 0$ always has one solution $X = 0$ it is called the trivial solution. It can have infinitely many solutions iff it has at least one non trivial (i.e. non zero) solution w. The equation $Aw = 0$ for a non trivial w implies that the columns of A are linearly dependent. Thus, the necessary and sufficient condition for infinitely many solutions is that the rank of A is less than its number of columns ($colnum(A)$).
For our 3×3 matrix A, this means its rank is less than 3. The easiest calculation is to note that

$$\det(A) = 1\left((-3+2) - 2(-4-t)\right) - 1\left(-4(-3+t) - (-3)(-1-t)\right) + (-1)\left(-4(2) - (-3)(1)\right) = 10 + 10t.$$

Thus, the answer is that $t = -1$ is the only value giving infinitely many solutions.

You may also find the rank from an REF of A:
$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & -8-t \\ 0 & 0 & 2+2t \end{pmatrix}$. Clearly, this has rank less than 3 iff $2 + 2t = 0$, i.e. $t = -1$.

2. **Points: 3+3+4+5**

(a) Complete the definition:

A set of vectors $S = \{v_1, v_2, \cdots, v_r\}$ is said to be **linearly dependent** if :

There are scalars $a_1, a_2, \cdots, a_r$ such that $a_1v_1 + a_2v_2 + \cdots + a_rv_r = 0$ and at least one of $a_1, \cdots, a_r$ is non zero. In other words $AX = 0$ has at least one non trivial solution.

Alternatively, we have an equivalent definition which says: there is an s with $1 \leq s \leq r$ such that v_s is a linear combination of $v_1, \cdots, v_{s-1}$.

(b) Explain why the two columns of the matrix $A = \begin{bmatrix} 3 & 6 \\ -2 & -6 \\ -4 & -4 \end{bmatrix}$ **are linearly independent.** Be sure to justify your answer in terms of your definition.

In terms of our first definition, we need to solve $AX = 0$ and determine that we have only the trivial solution. If we take v_1, v_2 as the columns of A in order, then we are solving $a_1v_1 + a_2v_2 = 0$. This gives 3 equations $3a_1 + 6a_2 = 0, -2a_1 - 6a_2 = 0, -4a_1 - 4a_2 = 0$. This gives $a_1 = -2a_2, a_1 = -3a_2, a_1 = -a_2$ respectively. The first two already force $a_1 = 0 = a_2$.

If we use the second definition then our s is 1 or 2. If $s = 1$ then $v_1 = 0$ (since the definition implies v_1 is a linear combination of an empty set of vectors!)
If $s = 2$, then we must have $v_2 = a_1v_1$ for some a_1. But the three different rows give inconsistent values for a_1, namely $2, 3, 1$ respectively.
This definition can give a quicker result using determinants. If v_2 is a multiple of v_1, then all 2×2 subdeterminants of A must be zero! So, we simply need to find a non zero subdeterminant, e.g. the one formed by the top two rows gives $-18 + 12 = -6 \neq 0$!

(c) Determine a vector v in $\Re^3$ such that v is **not** in $Col(A)$. You must justify your claim.

The condition is equivalent to the three columns v_1, v_2, v being independent. It is easy to note this by using the second definition since $v \in Col(A)$ means the three vectors are dependent.

So, all we need is a convenient independent vector. Try $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. You can show the determinant of the resulting 3×3 matrix to be $(1)(-6) \neq 0$.
Alternatively, you can determine the rank by an easy row reduction.

(d) Determine if the set of three vectors consisting of your v together with the two columns of the matrix A **linearly dependent.**

We already arranged the vectors to be linearly independent!

3. **Points: 4+4+3+4**

You are given a matrix A and a REF M of the augmented matrix $A|I$.

Use these to answer the following questions.

$$A = \begin{bmatrix} -2 & -6 & -8 \\ 2 & 2 & 4 \\ 4 & 8 & 12 \end{bmatrix} \text{ and } M = \left[\begin{array}{ccc|ccc} -2 & -6 & -8 & 1 & 0 & 0 \\ 0 & -4 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

(a) Given any column vector $v \in \Re^3$ is it true that the equation $AX = v$ is consistent? Be sure to justify your claim.

Consider the vector $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The given REF of $(A|I)$ informs us that the REF of $(A|v)$ will be the matrix formed by dropping the columns numbered $4, 5$ from M. This produces an inconsistent third row equation $0 = 1$.

So, the answer is no! An alternate argument would be that $AX = v$ is consistent for all v iff all v are in $Col(A)$, i.e. $Col(A) = \Re^m$ where $rownum(A) = m$. In our situation, $m = 3$.

But this requires that $rank(A) = rownum(A) = 3$ and the left hand part of M shows the rank to be 2.

(b) In case your answer is "no", determine the consistency matrix G, i.e. a matrix G such that $AX = v$ is consistent if and only if $Gv = 0$.

Hint: You may find the answer in the matrix M. As established in the class notes, the matrix G is simply the matrix on the right hand side of the matrix of zero rows in the REF of $(A|I)$. So $G = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$.

(c) Complete the definition:

A set S of vectors in $\Re^n$ is a spanning set for $\Re^n$ if :

every $v \in \Re^n$ is a linear combination of some finite set of vectors from S.

(d) Determine, with proof, if the columns of A give a spanning set for $\Re^3$.

Be sure to use your definition.

We have already shown a vector v which is not a combination of the columns of A, since $AX = v$ is inconsistent. So, columns of A do not span $\Re^3$.

4. **Points: 3+4+5+3**

You are given a matrix A and its REF M. **Use these to answer the following questions.**

$$A = \begin{bmatrix} -1 & -2 & -3 & -1 \\ -2 & 3 & 1 & -9 \\ 3 & 2 & 5 & 7 \end{bmatrix} \text{ its REF } M = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Complete this definition:

Given a matrix H, its null space $Nul(H)$ is defined to be set of all vectors such that

$Hv = 0$, if the vector be named as v. Alternatively, $Nul(H) = \{v \mid Hv = 0.\}$

(b) **Using the given information,** determine $Nul(A)$ completely.

Suppose that $v = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. The system $Av = 0$ gives the following three equations by using the REF. $a + c + 3d = 0, b + c - d = 0, 0 = 0$.
This gives the two pivot variables a, b in terms of the free variables c, d. Hence we have

$$\begin{pmatrix} 1 \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -c - 3d \\ -c + d \\ c \\ d \end{pmatrix} = c \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let v, w be the vectors multiplied by c, d above. This shows that $Nul(A) = Span\{v, w\}$.

(c) Determine a matrix P **with four rows and two linearly independent columns** such that $AP = 0$. You should justify why your answer is correct by using the above calculations.

The two columns v, w will form the matrix P, giving $AP = 0$. The vectors v, w are seen to be independent since the resulting matrix has rank 2. This can be proved by noting the non zero 2 determinant formed by the bottom two rows.

(d) Suppose that a matrix H is invertible. Determine if $Nul(H)$ has a nonzero vector. Justify your answer.

Let $G = H^{-1}$ so that $GH = HG = I$. If $v \in Nul(H)$, then $Hv = 0$. Multiplying by G on the left, we have $GHv = G0$ which simplifies to $Iv = 0$ i.e. $v = 0$. So, for an invertible H, its Null space has only the zero vector.

First Supplement Quiz

1. Define what is meant by "the set of vectors $\{v_1, v_2, v_3\}$ are dependent."

   Answer: $\{v_1, v_2, v_3\}$ are said to be dependent if there are scalars a_1, a_2, a_3 such that $a_1v_1 + a_2v_2 + a_3v_3 = 0$ and at least one of a_1, a_2, a_3 is non zero.

2. Give examples of three **nonzero** vectors v_1, v_2, v_3 in $\Re^4$ such that they are linearly dependent, but none of the three vectors is a multiple of another.

   Answer: Let $v_1 = e_1, v_2 = e_2$ the two independent vectors of the standard basis of $\Re^4$. Let $v_3 = e_1 + e_2 = v_1 + v_2$ which gives the required dependence. By being a part of a basis, we know that v_1, v_2 are non zero and not multiples of each other. We claim that $v_3 \neq ke_1$ for any scalar k since that would give $e_1 + e_2 = ke_1$ or $(1-k)e_1 + e_2 = 0$, making e_1, e_2 dependent, contrary to the assumption. similarly v_3 is not a multiple of e_2. Alternatively, write down any two non zero v_1, v_2 which are not multiples of each other and then $v_3 = v_1 + v_2$ can be easily checked to satisfy all conditions by finding non zero 2×2 determinants in every two of the three columns.

3. Give examples of three **nonzero** vectors v_1, v_2, v_3 in $\Re^4$ which are linearly independent.

   Answer: Any three elements of the standard basis, say $\{e_1, e_2, e_3\}$ will be independent.

Second Supplement Quiz

1. Define what is meant by "a linear system $AX = B$ is consistent."

   Answer: A linear system $(A|B)$ is consistent if there is some vector v such that $Av = B$.

2. When is a linear system $AX = B$ said to be homogeneous? Prove that a homogeneous linear system is always consistent.

   Answer: A linear system $(A|B)$ is said to be homogeneous if B is a zero vector. Since $A0 = 0$, we can take $v = 0$ in the consistency definition for a homogeneous system.

3. Give an example of "a linear system of three equations in three variables x, y, z which is **inconsistent.**" Be sure to explain why it is inconsistent.

   Answer: A simplest example would be $(A|B) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ If $Av = B$ for some v, then the bottom entry of Av is zero, but it is 1 for B. Hence it is inconsistent.

4. Determine all values of t for which the following system has more than one solution. Show your argument.

x	y	z	RHS
1	0	1	0
-1	1	$2-t$	0
2	t	$t+2$	0

   Answer: A homogeneous system $(A|0)$ for a matrix A has only a trivial solution when $rank(A) = colnum(A)$. Thus we need values of t for which $rank(A) < 3$ or $\det(A) = 0$. The value of the determinant is $(1)(t + 2 - t(2 - t) + 0 + 1(-t - 2) = t^2 - 2t$ if we expand by the first row. Hence $t = 0, t = 2$ are the desired values.

Third Supplement Quiz

You are given the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & -2 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \text{ and its REF } M = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Determine the $Nul(A)$ using the given information. The answer must be in the usual parametric form. Show your work.

   Answer: We know that $Nul(A) = Nul(M)$ since A and M are related by row transformations. Explicitly $M = HA$ for some invertible H. Thus $AX = 0$ iff $HAX = 0$ iff $MX = 0$. Now, M gives $-y - z = 0$ and $x + y + 2z - w = 0$ by using x, y, z, w in order as the variables for columns of M. If we set $z = s$ and $w = t$, then we get the parametric solution: $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. The multipliers of s, t give a basis of two independent vectors for $Nul(A)$.

2. Determine if the linear system $(A|B)$ is consistent for all $B \in \Re^4$. Prove your claim.

   Answer: The answer is no. Since $rank(A) = 2$ as visible in REF, the space $Col(A)$ has dimension $2 < 4 = \dim \Re^4$. So $Col(A) \neq \Re^4$.

3. Set L to be the linear transformation from $\Re^n$ to $\Re^m$ defined by the formula $L(X) = AX$.

   From the shape of A, decide the values of n, m.

   $n =$ $m =$.

   Answer: $n = m = 4$.

   Further, decide if L is injective. Prove your claim.

   Answer: No! The condition for injectivity of T_A is that $Nul(A) = 0$. We showed this to be non zero above.

   Also decide if L is surjective. Prove your claim.

   Answer: No. The surjectivity of T_A requires that $Image(T_A) = Col(A) = \Re^4$, the codomain. Since $Col(A)$ has dimension 2 and $\Re^4$ has dimension 4, T_A is not surjective.