DEPARTMENT OF MATHEMATICS

Ma 322 Second Exam October 31, 2016

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Instructions: Be sure that your name, section number, and GUID are filled in below.
Calculators are permitted, so long as they are not capable of wireless communication.
Put your answers in the answer spaces provided. You must have supporting work in addition to answers.
It is extremely important to write clearly with a medium or softer pencil or black ink.
Do not write too close to the page borders.
The exam has six pages in addition to the front page and last two blank pages. If you need extra space for your answers, they may be put on the last two blank pages. However, it is essential that you indicate their location on the problem page. You may use the backs of exam pages for scratch work which need not be graded.

Problem	Maximum Score	Actual Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
Total	90	

The 90 points on the exam will be scaled to 100 in grade calculation.

Please fill in the information below.

NAME: ______________________________ **Section:** ____________

GUID: ________________

1. **Points: 6+4+5**

(a) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & -2 \\ -2 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix}.$$

Find the adjoint of A. It is necessary to write out the individual entries as cofactors to show evidence of using the formula.

(b) Use your calculations to find the determinants of the the matrices A and its adjoint $adj(A)$.

(c) Use Cramer's rule to solve the matrix equation:

$$\begin{bmatrix} 2 & 0 & -2 \\ -2 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix} X = \begin{vmatrix} -1 \\ 1 \\ -2 \end{vmatrix}$$

Note that the answer should be the three entries of the column vector X.

It is permissible to leave your answer as ratios of unevaluated determinants

2. **Points: 6+4+5**

For every value of the parameter t, define a linear transformation

$$L : P_2 \to P_2 \text{ by the formula } L(f(X)) = X^2 f''(X) - t f(X).$$

Recall that all $f(X)$ in P_2 are polynomials of degree at most 2

Answer the following questions about L.

(a) Using the standard basis $B = \begin{bmatrix} 1 & X & X^2 \end{bmatrix}$ for both domain and target P_2, determine the matrix of transformation for L

(b) Set $t = 1$ and determine if the transformation L is injective and/or surjective

Injective? ____________Surjective____________

Be sure to explain your reasoning!

(c) Determine all values of t for which the transformation L **is not injective**

Be sure to justify your answer, since unjustified answers will get no credit.

3. **Points: 3+4+6+2**

Let V be a four dimensional vector space over the real numbers with basis $B = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$

Using this information, answer the following questions.

(a) Let $W = Span\{w_1, w_2\}$ where $w_1 = v_1 + v_2$ and $w_2 = v_2 + v_3$

Briefly explain why W is a subspace of V and find the dimension of W.

(b) Define a linear transformation $T : V \to W$ defined by setting

$T(v_1) = w_1$, $T(v_2) = w_2$, $T(v_3) = w_1 - 4(w_2)$, $T(v_4) = w_1 + 2(w_2)$

Is T surjective? Be sure to justify your assertion.

(c) Choose a vector w_3 **and let** $w_4 = v_4$ such that $C = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}$ is also a basis of V.

$w_3 =$

Hint: Write $C = BP$ and argue that P is nonsingular. Be sure to explain why the nonsingularity guarantees that C is also a basis.

(d) Determine $[v_3]_C$ the coordinate vector of v_3 with respect to the basis C.

$[v_3]_C =$

4. **Points: 2+7+2+2+2**

(a) Let T denote a linear transformation from a vector space V to a vector space W. **Precisely state the Fundamental Theorem of Dimensions** which gives the relation between the dimension of V, dimension of $Ker(T)$ and dimension of $Image(T)$.

Note: For convenience, you may assume that V is finite dimensional.

(b) Let V be the vector space spanned by $v_1 = 1$,$v_2 = x$, $v_3 = \sin(4\,x)$, $v_4 = \cos(4\,x)$. Let T denote the linear transformation on V defined by the formula

$$T\left(f\left(x\right)\right) = f''(x) + 16\,f\left(x\right) \text{ for every } f(x) \text{ in } V\ .$$

Determine the images of all the generators of V.

$T\left(v_1\right) =$ ____________________ $T\left(v_2\right) =$ ____________________

$T\left(v_3\right) =$ ____________________ $T\left(v_4\right) =$ ____________________

(c) Find generators for $Image(T)$ and deduce the dimension of $Image(T)$.

(d) Find generators for $Ker(T)$ and deduce the dimension of $Ker(T)$.

(e) Using the above Fundamental Theorem or otherwise, prove that v_1, v_2, v_3, v_4 are linearly independent.

5. **Points: 3+3+3+3+3**

Let $A = \begin{bmatrix} -1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(a) Find a number α such that $Av_1 = \alpha v_1$ where $v_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

(b) Find a number β such that $Av_2 = \beta v_2$ where $v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

(c) Find a number γ such that $Av_3 = \gamma v_3$ where $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) Explain why $B = [v_1, v_2, v_3]$ is a basis of $\Re^3$.

(e) If T is the linear transformation $T(X) = AX$, then determine the matrix of T in the basis B.

Hint: To save work, recall the definition of the matrix of a transformation in a given basis.

6. **Points: 3+3+3+3+3**

We shall denote the vector space of all polynomials in a variable X with real coefficients by P. Answer the following questions. When you are asked for examples, be sure to use suitable subspaces of P.

(a) Define what is meant **by the dimension of a vector space** V over a field K.

(b) What is the dimension of the space $Span\{X, X^{12}, X^{32}\}$? Justify your answer.

(c) Define what is meant by saying that a sequence of n vectors $s = (v_1, v_2, \cdots, v_n)$ is **linearly dependent.**

(d) Suppose that you have a sequence of four polynomials $S = (f_1, f_2, f_3, f_4)$ each of degree at most 2. Either prove that S is linearly dependent, or produce four specific polynomials of degree at most 2, which are linearly independent.

(e) Let W be the set of all polynomials $f(X)$ in P such that:

(i) Each polynomial $f(X)$ has degree at most 4 and

(ii)$f(0) = 0$

Find a basis of W. What is the dimension of W?

7. **Extra work page 1: This page must not be removed!**

   This page is to be used to submit work which did not fit on the problem pages 1 to 6.

   Important: You must have clear reference to work displayed here. Without that, this page will be ignored.

8. **Extra work page 2: This page must not be removed!**

   This page is to be used to submit work which did not fit on the problem pages 1 to 6.

   Important: You must have clear reference to work displayed here. Without that, this page will be ignored.

1 Answer Key for exam2v3

1. $\diamond$ $\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$
   $\diamond$ $det(A) = 2$ and $det(adj(A)) = 4$
   $\diamond$ $\frac{-7}{2}, 0, -3$

2. $\diamond$ $\begin{bmatrix} -t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 2-t \end{bmatrix}$
   $\diamond$ Both injective and surjective since the determinant of M is non zero and hence has rank 3.
   $\diamond$ The matrix M has determinant $t^2\,(2-t)$
   and so it is nonsingular exactly when $t \neq 0$ and $t \neq 2$.
   Hence the answer is all real t other than $0, 2$.

3. $\diamond$ Since W is a span of a set of vectors in V, it is a subspace of V
   The nonzero vectors w_1 and w_2 are independent since neither is a multiple of the other since vectors in B are linearly independent.
   $\diamond$ The image space contains a spaning set for W, namely $\{w_1, w_2\}$ and hence equals W. Hence it is surjective.
   $\diamond$ Choose $w_3 = v_3$. We claim that the span of C contains all vectors of B.
   $w_2 - w_3 = v_2$ and $w_1 - v_2 = w_1 - w_2 + w_3 = v_1$
   Since $v_3 = w_3$ and $v_4 = w_4$, our claim is proved.
   Since we have four spanning vectors w_1, w_2, w_3, w_4 for the 4-dimensional space V they give a basis.
   $\diamond$ For our choice of $v_3 = w_3$, we simply get the column with entries $0, 0, 1, 0$.

4. $\diamond$ Let n denote the dimension of V and let d and e be respectively the dimensions of $Ker(T)$ and $Image(T)$. Then we have $n = d + e$.
   $\diamond$ The four images are $16, 16\,x, 0, 0$
   $\diamond$ $Image(T)$ is generated by 1 and x and thus has dimension 2, since these are independent.
   $\diamond$ The functions v_3 and v_4 map to 0 and these are known to be independent.
   Hence the dimension is 2.
   $\diamond$ The Fundamental Theorem gives that $\dim(V) = 2 + 2 = 4$. Since V is spanned by these four vectors, they must be a basis, hence independent.

5. ◇ 1
   ◇ -1
   ◇ 2
   ◇ Use the determinant, or rank.
   ◇ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

6.
   ◇ If V is spanned by a finite set of elements, then it is said to have finite dimension. In this case, its dimension is the number of elements in any **spanning set of independent vectors**
   If no finite spanning set exists, then V has infinite dimension.
   ◇ Since it has a 3 element spanning set, the dimension is at most 3.
   Moreover, the three monomials are linearly independent by definition of polynomials,
   or by being a part of known basis of P_{32} So the dimension is 3.
   ◇ The sequence is said to be linearly dependent if there are scalars $a_1, a_2, \ldots, a_n$ such that $\sum_1^n a_i v_i = 0$ and **at least one of a_i is nonzero.**
   ◇ Since P_2 has a spanning set of three vectors $1, x, x^2$,its dimension is at most 3.
   Since S has 4 elements, they are linearly dependent, lest P_2 will have dimension at least 4.
   ◇ The condition $f(0) = 0$ implies that the polynomials are all divisible by x.
   Hence the space W is spanned by x, x^2, x^3, x^4 due to the degree condition.
   These are known to be independent, so dimension is 4.

7.

8.