DEPARTMENT OF MATHEMATICS

Ma 322 Second Exam October 31, 2016

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Instructions: Be sure that your name, section number, and GUID are filled in below.
Calculators are permitted, so long as they are not capable of wireless communication.
Put your answers in the answer spaces provided. You must have supporting work in addition to answers.
The exam has six pages in addition to the front page and last two blank pages. If you need extra space for your answers, they may be put on the last two blank pages. However, it is essential that you indicate their location on the problem page. You may use the backs of exam pages for scratch work which need not be graded.

Problem	**Maximum Score**	**Actual Score**
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
Total	90	

The 90 points on the exam will be scaled to 100 in grade calculation.

Please fill in the information below.

NAME: ______________________________ **Section:** ____________

GUID: __________________

1. **Points: 6+4+5**

(a) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & 1 & -3 \\ 24 & 4 & -11 \end{bmatrix}.$$

Determine the adjoint of A.It is necessary to write out the individual entries as cofactors to show evidence of using the formula.

(b) Use your calculations to determine the determinants of the the matrices A and its adjoint A^{adj}.

(c) Use Cramer's rule to solve the equations

$$\begin{bmatrix} 2 & 0 & -1 \\ 6 & 1 & -3 \\ 24 & 4 & -11 \end{bmatrix} X = \begin{vmatrix} 3 \\ 4 \\ -1 \end{vmatrix}$$

It is permissible to leave your answer as ratios of unevaluated determinants

2. **Points: 3+3+4+5**

Define a linear transformation

$$L : P_2 \to P_2 \text{ by the formula } L(f(X)) = X^2 f''(X) - t f(X).$$

where t is a parameter

Answer the following questions about L.

(a) Using the standard basis $B = \begin{bmatrix} 1 & X & X^2 \end{bmatrix}$ for both domain and target P_2, determine the matrix of transformation for L

(b) Set $t = 3$ and determine if the transformation L is injective and/or surjective

Injective? ____________Surjective____________

Be sure to explain your reasoning!

(c) Determine all values of t for which the transformation L is injective.

(d) Determine **with proof** all values of t for which **the transformation L is not injective.**

3. **Points: 4+4+3+4**

Let V be a four dimentional vector space over real numbers with basis
$B = \left[\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \right]$

Using this information, answer the following questions.

(a) Let $W = Span\{w_1, w_2\}$ where $W_1 = v_1 + v_2$ and $w_2 = v_2 + v_3$

Briefly explain why W is a subspace of V and determine its dimension.

(b) Define a linear transformation $T : V \to W definedbysetting$

$T(v_1) = w_1$, $T(v_2) = w_2$, $T(v_3) = w_1 + 3$, $T(v_4) = w_1 - 3(w_2)$

Determine, with proof, if T is surjective.

(c) Choose a vector w_3 **and let** $w_4 = v_4$ such that $C = \left[\begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \right]$ is also a basis ov V.

$w_3 =$____________

Hint: Write $C = BP$ and argue that P is non singular. Be sure to explain why this guarantees that C is also a basis.

(d) Determine $[v_3]_C$ the coordinate vector of v_3 with respect to the basis C.

$[v_3]_C =$____________

4. **Points: 4+4+3+4**

(a) **Precisely state the Fundamental Theorem of Dimensions** which gives the relation between the dimensions of the kernel and the image of a linear transformation T on a vector space V and the dimension of V.

(b) Let V be the vector space spanned by $v_1 = 1$,$v_2 = x$, $v_3 = \sin(2\,x)$, $v_4 = \cos(2\,x)$. Let T denote the linear transformation on V defined by the formula

$$T\,(f\,(x)) = f''x + 4\,f\,(x) \text{ for every } f(x) \text{ in } V\ .$$

Determine the images of all the generators of V.

$T\,(v_1) =$__________________ $T\,(v_2) =$__________________

$T\,(v_3) =$__________________ $T\,(v_4) =$__________________

(c) Determine the $Image(T)$ and deduce its dimension.

(d) Determine the $Ker(T)$ and deduce its dimension.

(e) Use the theorem to deduce that v_1, v_2, v_3, v_4 form a basis of V

5. **Points: 4+4+3+4**

Let $A = \begin{bmatrix} -14 & 24 & 0 \\ -8 & 14 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Determine a number α such that $Av_1 = \alpha v_1$ where $v_1 = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$

(b) Determine a number β such that $Av_2 = \beta v_2$ where $v_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

(c) Determine a number γ such that $Av_3 = \gamma v_3$ where $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) Explain why $B = [v_1 v_2 v_3]$ is a basis of $\Re^3$.

(e) If T is the linear transformation $T(X) = AX$, then determine the matrix of T in the basis B.

Hint: To save work, recall the definition of the matrix of a transformation in a given basis.

6. **Points: 4+5+6**

We shall denote the set of all polynomials in a variable X with real coefficients by P.Answer the following questions. When you are asked for examples, be sure to use suitable subspaces of P.

(a) Define what is meant **by the dimension of a vector space** V over a field K.

(b) What is the dimension of the space $Span\{X, X^9, X^{18},\}$? Justify your answer.

(c) Define what is meant by saying that a sequence of n vectors $s = (v_1, v_2, \cdots, v_n)$ is **linearly dependent.**

(d) Suppose that you have a sequence of four polynomials $S = (f_1, f_2, f_3, f_4)$ of degree at most 2 in P. Either prove that the S is linearly dependent, or produce four concrete polynomials satisfying the condition, which are linearly independent.

(e) Let W be the set of all polynomials $f(X)$ in P which satisfy the following conditions:

(i)$\deg(f(X)) \leq 4$ and

(ii)$f(0) = 0$

Determine the dimension of W by giving an explicit basis for W.

1 Answer Key for exam2vtry

1. ◇ $\begin{bmatrix} 1 & -4 & 1 \\ -6 & 2 & 0 \\ 0 & -8 & 2 \end{bmatrix}$

   ◇ $det(A) = 2anddet(A^{adj}) = 4$

   ◇ ¡ -7,-5,-17¿

2. ◇ $\begin{bmatrix} -t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 2-t \end{bmatrix}$

   ◇ Both injective and surjective since the determinant of M is non zero and hence has rank 3.

   ◇ The matrix M has determinant $t^2\,(2-t)$
   and so it is nonsingular exactly when $t \neq 0$ and $t \neq 2$. Hence the answer is all real t other than $0, 2$

   ◇ As seen above the non-injective values of t are $0, 2$.

3. ◇ *ans1*

   ◇ *ans2*

   ◇ *ans3*

   ◇ *ans4*

4. ◇ *ans1*

   ◇ *ans2*

   ◇ *ans3*

   ◇ *ans4*

   ◇ *ans5*

5. ◇ 2

   ◇ -2

   ◇ 1

   ◇ Use the determinant, or rank.

   ◇ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. ◇ *ans1*

   ◇ *ans2*

   ◇ *ans3*

   ◇ *ans4*

   ◇ *ans5*