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DEPARTMENT OF MATHEMATICS

Ma322005(Sathaye) - Final Exam Spring 2017

May 3, 2017

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Be sure to show all work and justify your answers.

There are 6 problems and a total of 9 pages including the front page and two pages at the end for extra work. No other sheets, books, papers are allowed.

Be sure to record any additional work on the last two pages saved for it. You may use backs of the main pages for scratch work, but this work will not be graded. If some work is saved on the last two pages, be sure to refer to it in the question on the main pages.

Problem	**Maximum Score**	**Actual Score**
1	16	
2	16	
3	16	
4	16	
5	16	
6	16	
Bonus	4	
Total	100	

Q.1: (2+2)+4+2+(3+3)

1. Let V and W be vector spaces defined over real numbers.

   Complete the following definitions and answer related questions.

   (a) A map L from V to W is defined to be a linear transformation if the following two conditions hold:

   - *Linearity.*

   - *Scalar multiplication.*

   Answer:

   (b) Explicitly describe a map $T : \Re^2 \to \Re^2$ such that T is **not a linear transformation**. You must explain which condition fails and how.

   Answer:

   (c) Suppose that $L : \Re^n \to \Re^d$ is defined by the formula $L(X) = AX$ where A is a 6×3 matrix. Determine the values of n and d with explanation.

   Answer:

(d) Write down a 3×3 matrix A **with all non zero entries** such that the linear transformation $T_A : \Re^3 \to \Re^3$ defined by $T_A(X) = AX$ is **not surjective** (onto).

You must explain why T_A is not surjective.

Answer:

Is it possible to construct the matrix A as described above such that the map T_A is injective but not surjective?

Either construct such a matrix A or explain why it cannot exist.

Answer:

Q.2: 5+5+3+3

1. Define a linear transformation $L : P_3 \to P_3$ by the formula $L(p(t)) = tp'(t) - cp(t)$ where c is an unspecified scalar.

   Thus, for example,

   $$L(1 + t + t^2 + t^3) = (3 - c)\, t^3 + (2 - c)\, t^2 + (1 - c)\, t - c.$$

   Answer the following questions about this transformation.

   (a) Recall the standard basis $B = \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix}$ for P_3.
   Calculate the **matrix of L in the basis B.**
   The matrix found shall be referred to as M in the following questions on this page.

   Answer:
   $M =$**Answer:**

   (b) Determine all the eigenvalues (which will be expressions involving c) for the matrix M.
   Moreover, for each eigenvalue, determine a corresponding eigenvector belonging to it.

   Answer:

   (c) Prove that L is diagonalizable (which is the same as M being diagonalizable.)

   Answer:

(d) Determine a value of c, if possible, for which the matrix M has only three different eigenvalues. Justify your claim.

Answer:

$c =$

Prove or disprove that L is still diagonalizable for the above chosen value of c.

Answer:

Q.3: 2+(2+2)+6+4

1. You are given two matrices:

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix},\ Q = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \text{ and told that } PQ = I.$$

Let v_1, v_2, v_3 be the three columns of P in order.

Use this information to answer the following questions.

(a) Assume that v_1 is an eigenvector for a 3×3 matrix A with eigenvalue 3. Calculate Av_1. The answer should be a vector in $\Re^3$.

Answer:
$Av_1 =$

(b) Also assume that v_2 is an eigenvector for the same A with eigenvalue -2 and v_3 similarly is an eigenvector for A with eigenvalue 1. Calculate Av_2, Av_3.

Answer:

$Av_2 =$ $\qquad\qquad$ $Av_3 =.$

(c) Determine the matrix A using the above information. Your final answer must be a single 3×3 matrix! You should first write A as a certain product of matrices and then the final answer is allowed to be from a calculator.

Answer:
$A =$**Answer:**

(d) Complete this definition:
A square matrix M is said to be diagonalizable if

Answer:

Prove that the above matrix A is diagonalizable by using your definition.

Answer:

Q.4: 4+3+3+3+3

1. Suppose that V is a vector space with a basis $B = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. Further suppose that V has an inner product with the inner product matrix:

$$< B, B >= \begin{pmatrix} 1 & -2 & 4 \\ -2 & 5 & -4 \\ 4 & -4 & 36 \end{pmatrix}.$$

Use this information to answer the following questions about vectors in V.

   (a) Determine the angle (in degrees) between the vectors v_1 and v_2 using the given inner product. **Set up the formula** before evaluation.

   Answer:

   (b) Set $w_1 = v_1$. Determine a scalar c such that w_1 and $w_2 = v_2 + cv_1$ are perpendicular under this inner product.

   Answer:

   $c =$ **Answer:**

   (c) Determine the projection of v_3 onto the space spanned by w_1.

   Hint: The answer is a vector $k_1 w_1$ such that $v_3 - k_1 w_1$ is perpendicular to w_1. Denote the projection vector by h_1.

   Answer:

   $h_1 =$ **Answer:**

   (d) Similarly determine the projection of v_3 onto the span of w_2 found above. Denote the new projection vector by h_2.

   Answer:

   $h_2 =$ **Answer:**

   (e) Set $w_3 = v_3 - h_1 - h_2$. From your calculations, prove that w_3 is perpendicular to w_1 as well as w_2.

   Answer:

Q.5: 2+(4+4)+2+2+2

1. Suppose that W is a vector space with a basis $C = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}$. Further suppose that V has an inner product with the inner product matrix:

$$< C, C >= H = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 5 & 3 & 0 \\ 0 & 3 & 10 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Work out the Gram-Schmidt algorithm as taught in the class notes by answering the following questions.

(a) Write out the augmented matrix $(H|I_4)$. Double check your entries.

Answer:

$(H|I_4) =$

(b) Carry out the REF algorithm on the above matrix $(H|I_4) =$. Only two steps are needed.

Warning: Beware of calculator answers. They may not be compatible with the current algorithm. Follow the strict algorithm as taught.

Answer:

(c) Using the REF, determine the matrix R which should be 4×4 unit upper triangular.

Answer:

$R =$ **Answer:**

(d) Report the new basis $C^* = CR$ which should be a 4 element row of linear expressions in w_1, w_2, w_3, w_4.

Answer:

$C^* =$

Answer:

(e) Finally determine the inner product matrix for the new basis C^*.

Hint: You should have a diagonal matrix and no new calculations are needed!

Answer:

$< C^*, C^* >$.

Answer:

Q.6: 6+2+4+2+2

1. Answer the following questions. In proving a result in some part, you may assume the above parts as proved.

   (a) Suppose that A is a 4×4 matrix with characteristic polynomial $f(\lambda) = \lambda^4 - 5\lambda^2 + 6$. Determine all the eigenvalues of A.

   Hint: The polynomial is a quadratic polynomial in λ^2.

   Answer:

   Eigenvalues are:

   (b) Prove that A is diagonalizable. Explain your argument.

   Answer:

   (c) Any square matrix P which satisfies the equation $P^2 - P = 0$ **is said to be a projection matrix.**

   Suppose that P is a projection matrix.

   If $v = Pw$ for some vector w, then prove that $Pv = v$.

   Hint: A matrix A is a zero matrix if and only if $Aw = 0$ for all w (of compatible size).

   Answer:

   (d) Prove that 1 and 0 are the only possible eigenvalues for a projection matrix P.

   Hint: Suppose that u is an eigenvector belonging to an eigenvalue c. Use the given condition to evaluate $(P^2 - P)u$.

   Answer:

   (e) Suppose that P is a non zero projection matrix. Use the above result to show that every non zero vector in $Col(P)$ is an eigenvector for eigenvalue 1.

   Answer:

   (f) **Extra credit 8 points:** *This extra credit work is best written on page 8.*

   Consider the linear transformation $T_P : \Re^n \to \Re^n$ where P is a projection matrix.

   - **Prove that** $Ker(T_P) = Nul(P)$ and $Image(T_P) = Col(P)$.
     Answer:
   - **Also prove that** $I - P$ is a projection matrix as well. How are its Kernel and Image related to those of P?
     Answer:
   - **Prove or disprove** Every projection matrix is diagonalizable!
     Answer:

Extra Page 1

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This page is to be used to submit work which did not fit on the problem pages 1 to 4.
Important: You must have clear reference to work displayed here.
Without that, this page will be ignored.

Extra Page 2

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