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DEPARTMENT OF MATHEMATICS

Ma322005(Sathaye) - Final Exam Spring 2017
May 3, 2017

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Be sure to show all work and justify your answers.
There are 6 problems and a total of 9 pages including the front page and two pages at the end for extra work. No other sheets, books, papers are allowed.

Be sure to record any additional work on the last two pages saved for it. You may use backs of the main pages for scratch work, but this work will not be graded. If some work is saved on the last two pages, be sure to refer to it in the question on the main pages.

Problem	**Maximum Score**	**Actual Score**
1	16	
2	16	
3	16	
4	16	
5	16	
6	16	
Bonus	4	
Total	100	

Q.1: (2+2)+4+2+(3+3)

1. Let V and W be vector spaces defined over real numbers.

   Complete the following definitions and answer related questions.

   (a) A map L from V to W is defined to be a linear transformation if the following two conditions hold:

   - *Linearity.*

   - *Scalar multiplication.*

   Answer: Linearity: $L(v_1 + v_2) = L(v_1) + L(v_2)$ and Scalar multiplication $L(cv_1) = cL(v_1)$ for all $v_1, v_2 \in V$ and scalars c.

   (b) Explicitly describe a map $T : \Re^2 \to \Re^2$ such that T is **not a linear transformation**. You must explain which condition fails and how.

   Answer: Many possible answers: Define $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ y \end{pmatrix}$. Both conditions fail when we take $v_1 = v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $c = 2$.

   (c) Suppose that $L : \Re^n \to \Re^d$ is defined by the formula $L(X) = AX$ where A is a 6×3 matrix. Determine the values of n and d with explanation.

   Answer: For AX to be defined, we need X to have 3 entries, so $n = 3$. Similarly, for AX to be in $\Re^d$ A must have d rows, so $d = 6$.

(d) Write down a 3×3 matrix A **with all non zero entries** such that the linear transformation $T_A : \Re^3 \to \Re^3$ defined by $T_A(X) = AX$ is **not surjective** (onto).

You must explain why T_A is not surjective.

Answer: The dimension of the image of T_A is the rank of A. So, we need a matrix A of rank less than 3. Take A of rank 1 given as $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$.

Is it possible to construct the matrix A as described above such that the map T_A is injective but not surjective?

Either construct such a matrix A or explain why it cannot exist.

Answer: This is not possible. The map T_A is injective iff the three columns of A are independent iff $rank(A) = 3$. Similarly It is surjective exactly when the number 3 of rows equals the $rank(A)$. Hence, injectivity holds iff surjectivity holds.

Q.2: 5+5+3+3

1. Define a linear transformation $L : P_3 \to P_3$ by the formula $L(p(t)) = tp'(t) - cp(t)$ where c is an unspecified scalar.

   Thus, for example,

   $$L(1 + t + t^2 + t^3) = (3 - c)\,t^3 + (2 - c)\,t^2 + (1 - c)\,t - c.$$

   Answer the following questions about this transformation.

   (a) Recall the standard basis $B = \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix}$ for P_3.
   Calculate the **matrix of L in the basis B.**
   The matrix found shall be referred to as M in the following questions on this page.

   Answer: The images of the basis vectors (polynomials) are $L(1) = t(0) - c(1) = -c$, $L(t) = t(1) - c(t) = (1 - c)t$, $L(t^2) = t(2t) - ct^2 = (2 - c)t^2$ and $L(t^3 = t(3t^2) - c(t^3) = (3 - c)t^3$. The corresponding coordinate vectors give the matrix $M =$**Answer:** $\begin{pmatrix} -c & 0 & 0 & 0 \\ 0 & 1-c & 0 & 0 \\ 0 & 0 & 2-c & 0 \\ 0 & 0 & 0 & 3-c \end{pmatrix}$.

   (b) Determine all the eigenvalues (which will be expressions involving c) for the matrix M.
   Moreover, for each eigenvalue, determine a corresponding eigenvector belonging to it.

   Answer: Since the matrix $M - \lambda I$ is diagonal, the eigenvalues are given by the diagonal entries $-c, 1 - c, 2 - c, 3 - c$. The corresponding eigenvectors are the four columns of I_4 by the voodoo principle.

(c) Prove that L is diagonalizable (which is the same as M being diagonalizable.)

Answer: First, since M is visibly diagonal, it is diagonalizable! You could also observe that R^4 has a basis of 4 eigenvectors of M, so L (or M) have *eigendim* 4. You could also use the fact that the four eigenvalues are distinct.

(d) Determine a value of c, if possible, for which the matrix M has only three different eigenvalues. Justify your claim.

Answer: This is not possible since the four eigenvalues are distinct for all values of c. $c =$

Prove or disprove that L is still diagonalizable for the above chosen value of c.

Answer: Yes, as shown above.

Q.3: 2+(2+2)+6+4

1. You are given two matrices:

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \text{ and told that } PQ = I.$$

Let v_1, v_2, v_3 be the three columns of P in order.

Use this information to answer the following questions.

(a) Assume that v_1 is an eigenvector for a 3×3 matrix A with eigenvalue 3. Calculate Av_1. The answer should be a vector in $\Re^3$.

Answer: $3v_1 = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$. $Av_1 =$

(b) Also assume that v_2 is an eigenvector for the same A with eigenvalue -2 and v_3 similarly is an eigenvector for A with eigenvalue 1. Calculate Av_2, Av_3.

Answer: Respectively, $-2v_2 = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$. $Av_2 =$ $Av_3 =$.

(c) Determine the matrix A using the above information. Your final answer must be a single 3×3 matrix! You should first write A as a certain product of matrices and then the final answer is allowed to be from a calculator.

Answer: Let $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $AP = PD$. Hence $A = PDP^{-1} = PDQ$. This gives $A =$**Answer:** $\begin{pmatrix} 5 & -4 & -1 \\ 7 & -6 & -1 \\ -4 & 4 & 2 \end{pmatrix}$.

(d) Complete this definition:

A square matrix M is said to be diagonalizable if

Answer: $HMH^{-1} = D$ where H is some invertible matrix and D is a diagonal matrix.
Prove that the above matrix A is diagonalizable by using your definition.

Answer: Note that $A = PDP^{-1}$ gives $P^{-1}AP = D$ or $QAP = D$.

Q.4: 4+3+3+3+3

1. Suppose that V is a vector space with a basis $B = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. Further suppose that V has an inner product with the inner product matrix:

$$< B, B >= \begin{pmatrix} 1 & -2 & 4 \\ -2 & 5 & -4 \\ 4 & -4 & 36 \end{pmatrix}.$$

Use this information to answer the following questions about vectors in V.

(a) Determine the angle (in degrees) between the vectors v_1 and v_2 using the given inner product. **Set up the formula** before evaluation.

Answer: The angle is given by

$$\theta = \arccos\left(\frac{< v_1, v_2 >}{\sqrt{< v_1, v_1 >< v_2, v_2 >}}\right).$$

Evaluation gives: $\arccos(\frac{-2}{\sqrt{1 \cdot 5}}) = 153.4349488°$.

(b) Set $w_1 = v_1$. Determine a scalar c such that w_1 and $w_2 = v_2 + cv_1$ are perpendicular under this inner product.

Answer: $0 =< w_1, w_2 >=< v_1, v_2 > +c < v_1, v_1 >= -2 + c$. So we get $c =$ **Answer:** $c = 2$.

(c) Determine the projection of v_3 onto the space spanned by w_1.

Hint: The answer is a vector $k_1 w_1$ such that $v_3 - k_1 w_1$ is perpendicular to w_1. Denote the projection vector by h_1.

Answer: As above, we solve for k_1 in $0 =< w_1, v_3 - k_1 w_1 >= 4 - k_1 \cdot 1$. Thus $k_1 = 4$ and we get: $h_1 =$ **Answer:** $4w_1$.

(d) Similarly determine the projection of v_3 onto the span of w_2 found above. Denote the new projection vector by h_2.

Answer: Similarly we solve $0 =< w_2, v_3 - k_2 w_2 >$ for k_2. The equation becomes $0 =< w_2, v_3 > -k_2 < w_2, w_2 >= 4 - k_2(1)$. So $k_2 = 4$. $h_2 =$ **Answer:** $4w_2$.

(e) Set $w_3 = v_3 - h_1 - h_2$. From your calculations, prove that w_3 is perpendicular to w_1 as well as w_2.

Answer: By construction, $v_3 - h_1$ is perpendicular to w_1 and $h_2 = k_2 w_2$ is clearly perpendicular to w_1, we see that w_3 is perpendicular to w_1. Similarly, observe that $< v_3 - h_2$ as well as h_1 are perpendicular to w_2 and so w_3 is perpendicular to w_2.

Q.5: 2+(4+4)+2+2+2

1. Suppose that W is a vector space with a basis $C = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}$. Further suppose that V has an inner product with the inner product matrix:

$$< C, C >= H = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 5 & 3 & 0 \\ 0 & 3 & 10 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Work out the Gram-Schmidt algorithm as taught in the class notes by answering the following questions.

(a) Write out the augmented matrix $(H|I_4)$. Double check your entries.

Answer:

$$\begin{pmatrix} 2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 10 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$(H|I_4) =$

(b) Carry out the REF algorithm on the above matrix $(H|I_4) =$. Only two steps are needed.

Warning: Beware of calculator answers. They may not be compatible with the current algorithm. Follow the strict algorithm as taught.

Answer:

$$\begin{pmatrix} 2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) Using the REF, determine the matrix R which should be 4×4 unit upper triangular.

Answer: This is obtained from transposing the matrix in the last four columns:

$R =$ **Answer:**

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) Report the new basis $C^* = CR$ which should be a 4 element row of linear expressions in w_1, w_2, w_3, w_4.

Answer: We get $C^* =$

Answer:

$$\begin{pmatrix} w_1 & -w_1 + w_2 & w_1 - w_2 + w_3 & w_4 \end{pmatrix}$$

(e) Finally determine the inner product matrix for the new basis C^*.

Hint: You should have a diagonal matrix and no new calculations are needed!

Answer: We simply pick up the diagonal of the final REF calculated above. $< C^*, C^* >$.

Answer:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Q.6: 6+2+4+2+2

1. Answer the following questions. In proving a result in some part, you may assume the above parts as proved.

   (a) Suppose that A is a 4×4 matrix with characteristic polynomial $f(\lambda) = \lambda^4 - 5\lambda^2 + 6$. Determine all the eigenvalues of A.
   Hint: The polynomial is a quadratic polynomial in λ^2.
   Answer: First, it factors as a polynomial in λ^2: $(\lambda^2 - 2)(\lambda^2 - 3)$. Further factorization of the quadratics gives 4 distinct eigenvalues: $\sqrt{2}, \sqrt{2}, -\sqrt{3}, \sqrt{3}$. Eigenvalues are:

   (b) Prove that A is diagonalizable. Explain your argument.
   Answer: Since we have $4 = Colnum(A)$ distinct eigenvalues, A is diagonalizable.

   (c) Any square matrix P which satisfies the equation $P^2 - P = 0$ **is said to be a projection matrix.**
   Suppose that P is a projection matrix.
   If $v = Pw$ for some vector w, then prove that $Pv = v$.
   Hint: A matrix A is a zero matrix if and only if $Aw = 0$ for all w (of compatible size).
   Answer: Since $v = Pw$ the needed equation becomes $P^2w = Pw$ or $(P^2 - P)w = 0$. This is true since $P^2 - P$ is zero.

   (d) Prove that 1 and 0 are the only possible eigenvalues for a projection matrix P.
   Hint: Suppose that u is an eigenvector belonging to an eigenvalue c. Use the given condition to evaluate $(P^2 - P)u$.
   Answer: By the hint, we get $0 = (P^2 - P)u = (c^2 - c)u$. Since $u \neq 0$, we have $c^2 = c$, i.e. $c = 0$ or $c = 1$.

   (e) Suppose that P is a non zero projection matrix. Use the above result to show that every non zero vector in $Col(P)$ is an eigenvector for eigenvalue 1.
   Answer: A vector v is in the $Col(P)$ iff $v = PX$ for some X. Then as calculated above, $Pv = P^2X = PX = v$. Hence every non zero v is an eigenvector belonging to eigenvalue 1.

   (f) **Extra credit 8 points:** *This extra credit work is best written on page 8.*
   Consider the linear transformation $T_P : \Re^n \to \Re^n$ where P is a projection matrix.

   - **Prove that** $Ker(T_P) = Nul(P)$ and $Image(T_P) = Col(P)$.
     Answer: This is immediate by definitions. $Ker(T_P)$ is the set of all vectors v such that $T_P(v) = Pv = 0$, so it is the space $Nul(P)$.
     Similarly $Image(T_P)$ is the set of all vectors w of the form $w = T_P(v) = Pv$. So clearly, this is exactly the space $Col(P)$.
   - **Also prove that** $I - P$ is a projection matrix as well. How are its Kernel and Image related to those of P?
     Answer: Set $Q = I - P$ and calculate $Q^2 - Q$. This gives $(I - P)(I - P) - (I - P)$ which simplifies to $I - P - P + P^2 - I + P = I - P - P + P - I + P = 0$ as needed.
   - **Prove or disprove** Every projection matrix is diagonalizable!
     Answer: Let us determine the eigenspaces of each of the possible eigenvalues 0 and 1. As shown above, The eigenspace of 1 is simply the $Col(P)$. So, its dimension is $rank(P)$. Similarly, the eigenspace of 0 is the $Nul(P)$ and its dimension is $Colnum(P) - rank(P)$. The sum of these dimensions is $Colnum(P)$ so P is diagonalizable.

Extra Page 1

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Extra Page 2

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