Ma 322 **Preparation for Exam 2.** **November 05 2018**

Note: We covered the material for the first three chapters during the first exam. We also covered basic ideas of subspaces of $\Re^n$.

In the second exam, along with the weekly quizzes, we shall cover additional topics from Chapter 4 and 5.

As we go through these topics, we will be refining and extending our ideas of solving linear equations in abstract vector spaces.

Please pay careful attention to the correct definitions and notations for the new ideas, since without them, nothing will make sense!

Material based on Chapter 4

1. Do review the process of finding the adjoint of a square matrix and its uses for finding the inverse.

2. Review all quizzes, homework problems and lecture notes, including handouts. **Remember** that by now, calculations can be done by using a calculator, provided you state what you calculated and any formulas that you may have used.

   It is crucial to reproduce precise definitions, know how the definitions can be verified as well as how and when they are not applicable. Thus, the information and thoughts are the main things that are going to be tested.

3. It is not enough to have the "right intention", but you need the right words, written down on paper. Small numerical mistakes may be forgiven, but errors of understanding or incomplete reasoning will not!

4. Be familiar with many usual vector spaces and the ideas of bases and coordinate vectors in them.

   Thus, you should know the usual $\Re^n$ very well. You should also study P_n, $M_{2,2}$ (or $M_{r,s}$ in general. The spaces of functions on an interval should also be kept in mind as an important example. The notes have many such examples described.

5. Also, keep in mind the examples of infinite dimensional spaces: P the space of all polynomials in one variable over $\Re$, $\Re[[x]]$ the space of power series in x over $\Re$ and $C[0,1]$ the space of continuous real valued functions on $[0,1]$. This last example has interesting subspaces of once, twice or infinitely many times differentiable functions.

   You should be aware that the usual theorems of finite dimensional spaces need to be reinterpreted in such spaces.

6. I am giving the kind of questions you should ask and answer. Practice this.

7. Which P_n has dimension 5? **Answer:** P_4. Why? **Answer:** Because it has a basis of five polynomials $1, x, x^2, x^3, x^4$. Make more examples of other types. Think yourself why these are independent (by definition of polynomials).

8. Give examples of subspaces of P_3. **Answer:** They can be of dimensions ranging from 0 to 4.

   For dimension 2, take $Span\{2 - x, x^2 + x^3\}$. They are independent (why?), so the dimension of the resulting space is 2.

   Consider $W = \{p(x) \in P_3 \mid p(1) = 0\}$. **Answer:** This has dimension 3. Two ways of proving this are possible.

One is to show that the coordinate vectors of polynomials in the standard basis $B = (1 \;\; x \;\; x^2 \;\; x^3)$ satisfy the equation $a + b + c + d = 0$. in W. Solutions give a three dimensional subspace S of $\Re^4$ by a null space calculation. So our W is BS, hence also 3 dimensional.

Other method: By observation, $(x-1), x(x-1), x^2(x-1)$ are in W. These are seen to be independent by checking that the equation $a(x-1) + b(x)(x-1) + c(x^2)(x-1) = 0$ has only the trivial solution $a = b = c = 0$. So, the dimension is at least 3. It cannot be 4, since otherwise $W = P_3$ which is known to have dimension 4, but $1 \notin W$.

9. What are other ways of making subspaces? **Answer:** As a kernel of a linear transformation. For example, define $L : P_3 \to \Re$ by $L(p(x)) = p(0)$. Then the above space W is its kernel.

   Never forget that this corresponds to the Null space of a matrix A, when the linear transformation is T_A. **What does $\mathbf{T_A}$ mean?**

   Other answer: As the image of a linear transformation. Define a linear transformation T : $\Re^3 \to P_3$ by $T(\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a(x-1) + b(x)(x-1) + b(x^2)(x-1)$. Then W is easily seen to be the image of T.

   What does this mean when the transformation is T_A? **Answer:** $Col(A)$.

10. What is the definition of a set of independent vectors? Give examples of one, two, three etc. independent vectors in various spaces with justification.

    Answer: The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are independent and in $M_{2,2}$. Prove independence from definition or by finding their coordinate vectors in a standard basis and proving independence using rank.

11. Give examples of dependent vectors in a given vector space. Typically, you need to be able to find non trivial examples. This means no vectors are allowed to be zero and no two vectors are multiples of each other. For example, to get three matrices in $M_{2,2}$, we could take

    $$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

    The relation is $A - B + C = 0$.

12. What is meant by the span of a set of vectors? What is meant by the statement that a set of vectors is a spanning set of a vector space?

    See examples above. Make more of your own.

13. Give example of a spanning set for a vector space. Give example of a set of five vectors in a vector space V which is not a spanning set for V. Justify your claim.

    Answer; Take $V = P_2$. Try $1 + x, x + x^2, 1 + 2x + x^2, 1 + 3x + 2x^2, 4 + 5x + x^2$.

    What is a basis of the space spanned by them? **Note:** I made them by making combination of the first two, which were independent. Hence it is two dimensional, so does not span P_2.

14. When is a set of vectors said to be a basis of a vector space? **Recall:** definition.

15. Construct an examples of a vector space V the following examples in it.
    - A set of five vectors which forms a basis of V.
    - A set of five vectors which does not form a basis for V. (These five vectors should be non zero and none of them should be a multiple of another.)
    - A set of five vectors which do not span V. (These five vectors should be non zero and none of them should be a multiple of another.)
    - Prove that your vector space V cannot have six linearly independent vectors.
16. What is meant by the dimension of a vector space?
17. Give three **distinct examples** of an infinite dimensional space.
18. What is the definition of a linear transformation (homomorphism) of vector spaces?
19. Give three examples of linear transformations from a given vector space V to another given vector space W.

    Typically, to define a linear transformation from V to W, choose a basis B of V and for each basis vector choose **any desired** image in W. Extend this by linearity. **Study what this means!**

    This defines a linear transformation from V to W. Find its matrix by choosing a basis in W.

    For example: To define a linear transformation from P_3 to P_1 we choose bases $B = (1\ \ x\ \ x^2\ \ x^3)$ for P_2 and $C = (1\ \ x)$ for P_1.

    Define $T(1) = 1 - x, T(x) = 1 - 2x, T(x^2) = 2 - 3x, T(x^3) = 4x$.

    Verify that this has the matrix $A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & -2 & -3 & 4 \end{pmatrix}$.

    Clearly, $rank(A) = 2$, so dimension of the image is 2. Since the image is a subspace of the 2-dimensional space P_1, it is equal to the image, or T is surjective.

    By the fundamental dimension formula, $\dim(P_2) = 4 = \dim(Ker(T)) + \dim(Image(T)) = \dim(Ker(T)) + 2$.

    Hence $\dim(Ker(T)) = 2$. What is a basis?

    Check the $Nul(A)$ to have a basis $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \\ 0 \\ 1 \end{pmatrix}$. Thus, a basis for $Ker(T)$ is,

    $-1 - x + x^2, -4 + 4x + x^3$.
20. How do you determine if a linear transformation is injective? Give an example of the process.
21. How do you decide if a linear transformation is surjective? Give an example of the process.
22. What is meant by the Kernel and the Image of a linear transformation? Give at least three examples of linear transformations whose kernels and images are non zero vector spaces.
23. What are the two basic ways of describing a subspace of a vector space? (Answer: As a kernel of a linear transformation or as the image of a linear transformation.

    See examples above.

24. What is the fundamental dimension formula in vector spaces which relates the dimension of a vector space V, the dimension of the kernel of a linear transformation of V and the dimension of the image of the same transformation?

    Remember?

25. The fundamental dimension formula is related to the number of pivots in the row echelon form of a matrix. What is this connection?

26. If any independent set of vectors is given in a vector space, it can be enlarged to make a basis for the same vector space. Give an example of this process, where at least two extra vectors are needed.

    See above.

    Think: If the space has a finite dimension, then this works. If it has an infinite dimension, then we need to think some more.

    For example, in the space P of all polynomials with real coefficients, we can start with $1, x$ and we can happily add in $x^2, x^4, x^6, \cdots, x^{2n}, \cdots$. Even after adding infinitely many such vectors, we still have not spanned P!

    How can we resolve this?

27. Given any spanning set of a vector space, it can be trimmed down to a basis. Give an example of this process where at least two vectors need to be trimmed.

    See above.

28. Given a vector space V and an ordered basis $B = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ what is the definition of the coordinate vector $[v]_B$?

    Give example of finding such a vector in a finite dimensional space. Give a similar example in an infinite dimensional space.

    See above.

29. Given a linear transformation $T : V \to W$ and ordered bases B of V and C of W how do you find the matrix of the transformation with respect to these given bases?

    Answer: It is the matrix M whose columns are $[T(v_1)]_C, [T(v_2)]_C, \cdots, [T(v_m)]_C$ where $B = \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix}$.

    You may remember the formula as $[T(B)]_C$.

    See above.

    Give examples of such calculations when V is $M_{2,2}$. Same exercise when $V = P_n$ for some n. Corresponding W can be the same as V or different.

30. Given two ordered bases B and C of the same vector space V, how are they related by a matrix? How are the respective coordinates related by the same matrix?

    Answer: Write $C = BM$. Then $[v]_B = M[v]_C$ for all vectors v. The matrix M is suggestively denoted as P_B^C and its columns are simply the coordinate vectors of members of C with respect to the basis B.

    Give examples of such calculations in $\Re^n$. Do similar examples in other vectors spaces.

31. How does a determinant help in determining the rank of a matrix? **Answer;**

$$rank(A) = \max\{r \mid \ A \text{ has a non zero subdeterminant of size } r.\}.$$

Practice.

Material on Chapter 5

1. Learn the definitions of eigenvalues, eigenvectors and eigenspaces for a square matrix A.

   Learn to compute the sum of dimensions of all eigenspaces, i.e. $\sum_\lambda \dim(V(\lambda))$. Denote this sum by δ. Let us call it eigendim of A.

2. The diagonalization theorem states that a matrix A is diagonalizable iff its eigendim is equal to $n = rownum(A) = colnum(A)$.

More to come.