MA322
Sathaye **Final Preparations** **Fall 2018**

The final MA 322 exams will be given as described in the course web site (following the Registrar's listing). You should check and verify that you do not have a valid conflict with other final exams. If you have a conflict, you need to contact me at least 2 weeks in advance.

The questions on the final exam will be mainly on Chapters 4,5,6. Of course, all the calculations depend on the skills learned in the first three chapters, so you need to review them as well.

Expect about six questions based on the three chapters.

You should review the old exams, Webwork problems, daily quizzes and all the definitions. Important formulas should also be memorized to avoid serious mistakes caused by little flaws in recalling the formulas.

1. **Early Chapters with enhancements.**

   These topics involve the exam 1 material as well as its enhancement in the later periods.

   - Know what linear systems of equations are and how to relate them to augmented matrices. You should be able to carry out the REF/RREF calculations on a given augmented matrix without fail. Be sure to be able to handle matrices with variable entries.
   - Be prepared to write out the complete solution to a given linear system (after reduction to REF or RREF) including the notions of X_h - the homogeneous solution and X_p a particular solution.
   - Study the vector space interpretations of the solution process. In particular given a system $(A|B)$ its solution is used to determine bases (and hence dimensions) of the spaces $Col(A), Nul(A)$.
   - The solution process also gives information about independence/dependence of column vectors and consistency (solvability) of linear systems. Review the $0, 1, \infty$ principle.
   - Study the consistency matrix obtained from an REF of $(A|I)$ and its uses for consistency of equations. It can also be effectively used to calculate the $Nul(A)$ more naturally. It is also used to find orthogonal complement of $Col(A)$ and helps in the orthogonal projections.
   - Review and practice matrix calculations. It is crucial to study the relation between row/column transformations and matrix multiplications (and indeed

the voodoo principle in general). Be sure to study calculation of inverses especially for small sizes and possibly including parameter entries.

Study formulas for $(PQ)^{-1}$ and $(PQ)^T$. Don't forget how to solve $(A|B)$ if A is invertible.

Remember the $adj(M)$ and its uses.

- Study the determinants and techniques to determine their values using Laplace expansions and row/column transformations. Learn how to recognize (when possible) if the determinant is zero.

  Study the determinantal rank of a matrix and remember that it is equal to the usual rank of the matrix. In fact, the rank of a matrix has three equivalent definitions - number of pivots in REF, size of the largest nonzero subdeterminant and dimension of the column space. Be sure to use whichever is convenient in a problem.

2. **Chapter 4.** This covers the abstract Vector Spaces, but includes earlier and later ideas when relevant.

   - Define $\Re^n$ to be the set of all column matrices with n real entries. Given any real matrix $A = A_{n\times m}$ the space $Col(A)$ is a subspace of $\Re^n$.

     Equations represented by $(A|B)$ correspond to determining if $B \in Col(A)$. The row space of A denoted by $Row(A) = Col(A^T)$. The space $Nul(A)$ is the space of all vectors perpendicular to $Row(A)$.

   - Inspired by these connections, we define an abstract vector space V over a field K which has a valid scalar multiplication (by K) and addition. Be sure to review various examples of vector spaces, including polynomials, power series, function spaces etc.

   - The concept of rank of a matrix gets transformed to the dimension of a vector space. Be aware that (unlike a matrix) a vector space may have finite or infinite dimension.

     Carefully study the concepts of linear dependence/independence of vectors and a basis of a given vector space. Don't forget that when V has a finite dimension n, then any basis has exactly n elements. Moreover, any independent set of vectors has at most n elements and any set of more than n elements is necessarily dependent.

   - Given any basis $B = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ of a vector space V, every vector in V can be written uniquely as BX where X is a column of scalars. We define the coordinate vector of v in basis B as $[v]_B$ and note that $v = B[v]_B$ is valid for any basis B of the vector space V.

- We study subspaces and their properties. Given a vector space V, a subspace of V is a non empty subset W which is closed under the same operations as in V. In general, a subspace is a span of a non empty subset of V and can have any dimension between 0 and $\dim(V)$.
- We now reinterpret our equation $AX = B$ as a map which send a vector X to a new vector $AX = B$. This gives rise to the concept of a linear transformation from a vector space V to a vector space $W = Col(A)$.

  The analysis of when the linear transformation is injective and when it is surjective follows.

  If we choose a basis $B = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ for V and a basis $C = \begin{pmatrix} w_1 & w_2 & \cdots & c_m \end{pmatrix}$ for W then the matrix of any linear transformation $T : V \to W$ is given by a matrix A so that $T(BX) = CAX$.

  The matrix A is equal to $\begin{pmatrix} [T(v_1)]_C & [T(v_2)]_C & \cdots & [T(v_n)]_C \end{pmatrix}$. It can be suggestively written as T_C^B, or the matrix of the transformation T from $V = Span(B)$ to $W = Span(C)$.
- This leads to reinterpretation of properties of the transformation T in terms of the matrix T_C^B. Study the details.
- Two fundamental formulas about dimensions must be memorized.

  Given a linear transformation $T : V \to W$ where $\dim(V)$ is finite, we have $\dim(V) = \dim(Ker(T)) + \dim(Image(T))$. This generalizes the calculation for a matrix $A = A_{m\times n}$ which states:

  $n = rank(A) + \text{ number of free variables }$ when we solve $AX = B$.

  Note that at this stage, $rank(A) = \dim(Col(A))$

  The second and equivalent formula is $\dim(W_1+W_2)+\dim(W_1 \cap W_2) = \dim(W_1)+\dim(W_2)$ for any two finite dimensional subspaces W_1, W_2 of V.

3. **Remaining Chapters.**

- We begin by revisiting the effect of change of basis on the coordinate vectors. If B, C are two bases of a vector space V, then for every $v \in V$, we have $v = B[v]_B = C[v]_C$ i.e. $C[B]_C[v]_B = C[v]_C$ or $[v]_C = [B]_C[v]_B$. Thus the transformation is given by the multiplication by $[B]_C$.

  Note that $[B]_C[C]_B = I$ i.e. $[B]_C = [C]_B^{-1}$.
- If T is a linear transformation from V to V and has a matrix A with respect to a basis B and M with respect to a basis C, then we deduce that $M = [C]_B A [B]_C$. If we denote the matrix $[C]_B$ by P, then the formula is $M = PAP^{-1}$.

  A matrix M is said to be similar to A if we can write $M = PAP^{-1}$ for some P. Note that the concept holds only for square matrices and with P invertible.

- **Diagonalization.** We investigate matrices A which are similar to diagonal matrices D. Such a matrix A is said to be diagonalizable.

  Note that $A = PDP^{-1}$ for some invertible P. Moreover, $AP = PD$.

  If v_i is the i-th column of P then $Av_i = d_i v_i$ where d_i is the i-th entry on the diagonal of D.

  This gives rise to the definition of an eigenvector v of A which satisfies $Av = \lambda v$ for some scalar λ and $v \neq 0$. λ is then an eigenvalue belonging to v and an eigenspace of λ is the space of all v such that $Av = \lambda v$.

- To find eigenvalues of A, we define the characteristic polynomial of A by $P(\lambda) = \det(A - \lambda I)$.

  This leads to a simple test of diagonalizability:

  $A = A_{n\times n}$ is diagonalizable iff $\Re^n$ has a basis of eigenvectors of A, iff the sum of dimensions of different eigenspaces of A equals n (also known as the $eigendim(A)$).

  One easy sufficient condition is when A has n distinct eigenvalues.

- Transformations defined by diagonalizable matrices are easier to understand and describe. We note that $A^m = PD^mP^{-1}$ and hence powers of A are easy to calculate. In particular, $\lim_{m\to\infty} A^m v$ can be determined easily.

- It is important to be able to prove when A is not diagonalizable. In particular, when the eigenvalues are not real, then the matrix is not diagonalizable over reals.

- For a 2×2 matrix A, study the three distinct types of matrices which are similar to A, namely, diagonal, triangular with repeated entries along the diagonal and rotation type matrices (complex eigenvalues).

- **Inner Products.** We discussed a generalized version of dot products in vector spaces. This gives different notions of perpendicularity and angles in general vector spaces and leads to practical applications.

  Study the examples of inner products defined in terms of integrals (in function spaces) and by a direct inner product matrix relative to a basis.

  You should be able to do calculations similar to the dot products (length, distance, angle etc.) with inner products.

- **Normal equations and best fit.**

  We started with solving $AX = B$ and discovered when the system is inconsistent. We finish the discussion by showing that there is always a "best fit" solution, namely a vector X such that $(AX - B)$ has as small a norm as possible in the chosen inner product.

This can be shown to be the solution of the normal equations $A^TAX = A^TB$. If we use a general inner product, then this becomes $< A, A > X =< A, B >$.

We show that the normal equations are always consistent and the solution X gives the desired "best fit ".

Learn how to do the "best fit" problems for a given set of data points.

- **Projections.**

  The best fit solution X of the equation $AX = B$ gives a projection of B into the column space of A.

  If we consider the linear transformation which send X to AX, then it is easy to show that $B - AX$ is orthogonal to all vectors in $Col(A)$. This gives us the idea of orthogonal projection with respect to the generalized inner product.

- In case $V = \Re^n$ and $(w_1, \cdots, w_r)$ is a not necessarily an orthogonal basis of a subspace W, then the projection matrix is found thus:

  Let A be the matrix with columns $w_1, \cdots, w_r$ and let $P = A(A^TA)^{-1}A^T$.

  Then P is the matrix of the orthogonal projection into W, i.e. the projection of any v is given by Pv.

- In case we have a vector space V and $(w_1, \cdots, w_r)$ is an orthogonal basis of a subspace W, then the projection formula is much simpler:

  $$P(v) = c_1w_1 + \cdots + c_rw_r \text{ where } c_i = \frac{< v, w_i >}{< w_i, w_i >} \text{ for all } i.$$

  You should derive the matrix of this transformation using some basis of V.

- It is naturally useful to have a procedure to transform a basis of W into an orthogonal basis.

  This is called the Gram-Schmidt algorithm. Learn the simpler version taught in class and notes.