

1. Let $B = (b_1 \ b_2)$ and $C = (c_1 \ c_2)$ be two bases of a vector space such that

$$b_1 = 4c_1 + c_2 \text{ and } b_2 = 7c_1 + 2c_2.$$

- Find the 2×2 matrix L such that $B = CL$. **Note that** L is what we call P_C^B .

Answer: Note that

$$b_1 = 4c_1 + c_2 = C \begin{pmatrix} 4 \\ 1 \end{pmatrix} \text{ and } b_2 = 7c_1 + 2c_2 = C \begin{pmatrix} 7 \\ 2 \end{pmatrix}.$$

Hence: $B = (b_1 \ b_2) = C \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$. So $L = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$.

- Calculate $M = L^{-1}$. **Note that** $C = BM$ and $M = P_B^C$.

The $\det(L) = (4)(2) - (7)(1) = 1$, so $M = \frac{1}{1} \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}$.

- Let $x = b_1 + 3b_2$. What is $[x]_B$?

Answer: Since $x = B \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ we have $[x]_B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

What is $[x]_C$? **Hint:** You could use L .

Answer: Since $[x]_C = P_C^B[x]_B = L \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ we get $[x]_C = \begin{pmatrix} (4+21) \\ (1+6) \end{pmatrix} = \begin{pmatrix} 25 \\ 7 \end{pmatrix}$.

- Let $y = 2c_1 + c_2$. What is $[y]_C$?

Just as above, we get $y = C \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so $[y]_C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

What is $[y]_B$? **Hint:** You could use M .

Answer: Using $[y]_B = P_B^C[y]_C = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} (4-7) \\ (-2+4) \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

2. Find the eigenvalues and the corresponding eigenspaces for the matrix $\begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$.

Your answer should be a list of two eigenvalues together with a 2×2 matrix with the two eigenvectors as the columns in the same order.

Answer: The characteristic polynomial is:

$$\begin{vmatrix} (6 - \lambda) & -4 \\ 3 & (-1 - \lambda) \end{vmatrix} = (6 - \lambda)(-1 - \lambda) - (-4)(3) = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2).$$

Thus the eigenvalues are 3, 2.

For $\lambda = 3$ we find

$$\text{Nul} \begin{pmatrix} (6 - 3) & -4 \\ 3 & (-1 - 3) \end{pmatrix} = \text{Nul} \begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix}$$

and by the usual observation, a basis for this is noted as $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Strong suggestion: Do verify that

$$(A - 3I) \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Similarly, for $\lambda = 2$, we calculate:

$$\text{Nul} \begin{pmatrix} (6 - 2) & -4 \\ 3 & (-1 - 2) \end{pmatrix} = \text{Nul} \begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix}$$

and by the usual observation, a basis for this is noted as $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Do double check as above.

Thus, the required matrix is: $P = \begin{pmatrix} 4 & 4 \\ 3 & 4 \end{pmatrix}$.

Important!

We observe that the original matrix A satisfies:

$$AP = P \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \text{ and hence } A = P \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$$

gives the diagonalization of A .

3. Find the characteristic equation and the eigenvalues of the given matrix.

$$A = \begin{pmatrix} 10 & 2 & -7 & 10 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 5 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Find a basis for the eigenspace of the largest eigenvalue. **Hint:** After plugging in the the largest eigenvalue in the augmented matrix $[(A - \lambda I)|0]$, solve for the Null space by back substitution, even though the matrix is not quite in REF!

Answer: The characteristic polynomial is:

$$\det \begin{pmatrix} 10 - \lambda & 2 & -7 & 10 \\ 0 & 2 - \lambda & 1 & 3 \\ 0 & 0 & 5 - \lambda & 12 \\ 0 & 0 & 0 & -3 - \lambda \end{pmatrix} = (10 - \lambda)(2 - \lambda)(5 - \lambda)(-3 - \lambda).$$

Therefore, the eigenvalues are 10, 2, 5, -3.

For the highest eigenvalue 10 we solve for the null space for $A - 10I$:

$$\left[\begin{array}{cccc|c} x & y & z & w & 0 \\ 0 & 2 & -7 & 10 & 0 \\ 0 & -8 & 1 & 3 & 0 \\ 0 & 0 & -5 & 12 & 0 \\ 0 & 0 & 0 & -13 & 0 \end{array} \right].$$

Using the hint, we solve the last three equations by back substitution.

$-13w = 0$, so $w = 0$.

$-5z + 12w = 0$, so now $z = 0$ also.

$-8y + z + 3w = 0$, so now $y = 0$ also.

The first equation is now satisfied leaving x free.

So the solution space is spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. This is the eigen vector for $\lambda = 10$.

Do verify

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 10 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$