DEPARTMENT OF MATHEMATICS

Ma322004 (Sathaye) - FINAL EXAM Fall 1999
December 14, 1999

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Be sure to show all work and justify your answers.
There are 8 problems and a total of 9 pages including this one. No other sheets, books, papers are allowed.

Problem	Maximum Score	Actual Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
Total	160	

NAME: ______________________________

SECTION NO: ______________________________

STUDENT #: ______________________________

1. (20 Points)

(a) Given matrices A, B, when can you say that B is the inverse of A? In other words, define the "inverse" of a matrix.

(b) Give example of a 3×3 matrix A **without any zero entries** which **does not** have an inverse.

(c) If C, D are inverses of the matrices A, B respectively, then calculate the following expressions. Your answer should be a product involving A, B, C, D.

$$(AB)^{-1} = \qquad , (AC)^{-1} = \qquad , (A^2BAB)^{-1} =$$

(d) Define an elementary matrix. Give at least three eamples of elementary matrices.

(e) Write down a 3 by 3 matrix E such that $E \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$ for all $x, y, z \in \Re$.

2. (20 Points)

Write down a 4 by 7 matrix A which has exactly 3 nonzero pivots, after the usual Gauss elimination (if needed).

You have to answer many questions about it, so choose a matrix which will make your work easy. Think before using your pencil!

(a) What is the rank of the matrix A? How can you be sure?

(b) What is the dimension of $C(A)$? Write down an explicit basis for $C(A)$ Be sure to explain why your answer is a basis.

(c) Either prove that $C(A) = \Re^4$ or give a concrete vector in $\Re^4$ which is not in $C(A)$.

(d) Is there a vector B in $\Re^4$ such that $AX = B$ has no solution. Do explain your reasoning.

(e) What is a basis for $N(A)$? Explain why the vectors in your answer must be independent.

(f) What is a basis for $N(A^T)$? Again, explain why your answer is correct.

3. (20 Points)

This problem still belongs to your matrix A from the previous page. For our mutual convenience, copy the same matrix here.

(a) Suppose that T is a transformation from $\Re^7$ to $\Re^4$ given by the formula $T(v) = Av$. Is this a linear transformation? Why?

(b) Does the matrix A have an inverse? Why?

(c) Write down the **general solution** to the equation $AX = C_1 + C_2$, where C_1, C_2 are the first two columns of your matrix. Be sure to identify the usual X_p and X_n before writing the general answer.

4. (20 Points)

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ 0 & -1 \end{bmatrix}$. Answer the following questions about this matrix A.

(a) What is the dimension of $C(A)$? Why?

(b) What is a basis for $C(A^T)$?

(c) Find the projection matrix P for projecting into the column space of A. (Hint: $A(A^TA)^{-1}A^T$).

(d) Find the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ into $C(A)$.

(e) Does the equation $AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ have a solution? Why or why not?

(f) In your own words, explain the relation between the solution to $AX = B$ and the vector PB. Explain how this leads to "Least Squares Approximation". Write a small paragraph explaining this, using your calculations to illustrate your point.

5. (20 Points)

Suppose that A is a 3×3 matrix with three eigenvectors v_1, v_2, v_3 belonging to the eigenvalues $1, 1/2, 1/2$ respectively.

Assume further, that the vectors v_1, v_2, v_3 are respectively the columns of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) What is A? Simplify your answer.

(b) Express A in the diagonalized form, i.e. write it as $S\Lambda S^{-1}$.

(c) How is the diagonal form used to find $\lim_{n\to\infty} A^n$?

(d) Use the above information to evaluate the $\lim_{n\to\infty} A^n(v_1 + v_2 + v_3)$. Explain your reasoning. **(Hint: The answer is easy if you think about it, rather than blindly calculate.)**

6. (20 Points)

(a) Find all the eigenvalues and the corresponding eigenspaces for the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

Use your work to decide if A is diagonalizable. Explain all details of your reasoning.

(b) If M is a matrix satisfying the equation $M^2 = M$, show why $0, 1$ are the only possible eigenvalues for M.

(c) What are the eigenvalues and the eigenspaces for the 3×3 identity matrix I_3?

(d) What are the eigenvalues and the eigenspaces for the 3×3 zero matrix O?

7. (20 Points)

(a) T is a linear transformation from P_2 to P_2 given by the formula $T(f(X)) = f(x) + f'(x)$.

Find the matrix A of the transformation T in the bases G, G, where G is the basis $(1, x, x^2)$.

(b) An eigenvector of the transformation T is defined as a nonzero vector v such that $T(v) = \lambda v$ for some λ. The corresponding number λ is said to be an eigenvalue of T.

Explain why 1 is the only eigenvalue for T and find all eigenvectors belonging to 1. (Hint: connect your answer with eigenvectors and eigenvalues of A).

8. (20 Points)

A vector space V has basis $G = (v_1, v_2, v_3)$ and a linear transformation from V to itself is given by the formulas $T(v_1) = v_1 + v_2$, $T(v_2) = v_2 + v_3$, $T(v_3) = v_1 + v_2 + v_3$.

(a) What is the matrix of T in the bases G, G?

(b) If we change the basis G to the basis H given by $H = (v_1 + v_2, v_2, v_3)$, find the new matrix for T in the bases H, H. Be sure to simplify your answer.

(c) What are the dimensions of the Kernel and the Range of T? Show all work and reasoning.