DEPARTMENT OF MATHEMATICS

Ma322 - FINAL EXAM Fall 2010
December 17, 2010

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Be sure to show all work and justify your answers.
There are 7 problems and a total of 8 pages including this one. No other sheets, books, papers are allowed.

Problem	Maximum Score	Actual Score
1	14	
2	14	
3	14	
4	14	
5	14	
6	14	
7	14	
Free	2	
Total	100	

NAME: ______________________________

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ 5 & 5 & 5 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix}$$

This matrix will be use repeatedly in later questions as well.
Q.1.

1. Define what is meant by **the rank of a matrix. Definition.** An $m \times n$ matrix H is said to have rank d if $\cdots$

2. Define what is meant by a linear system of equations to be consistent.

   Definition. A linear system of equations $HX = Y$ is said to be consistent if $\cdots$

3. Now illustrate the above definitions using A as follows.

   - Calculate the rank of A.

   - Find a vector B such that $AX = B$ is **inconsistent.** Justify your answer.

   - Find another vector C such that $AX = C$ is **consistent.** Justify your answer.

Q.2.

1. Define what is meant by a linear transformation:

   Definition. Let V, W be vector spaces over $\Re$.

   A map $L : V \to W$ is said to be a linear transformation if $\cdots$

2. Let $L : V \to W$ be a linear transformation of real vector spaces. Define the following:

   - $Ker(L) = \left\{ v \in V \middle| \qquad\qquad\qquad\qquad \right\}.$
   - $Image(L) = \left\{ w \in W \middle| \qquad\qquad\qquad\qquad \right\}.$

3. Define a linear transformation $L : \Re^n \to \Re^m$ by the formula $L(X) = AX$ where A is the matrix introduced at the beginning.

   Answer the following questions.

   (a) because of the size of A, we must have $n =$ _____ and $m =$ _____

   (b) $Ker(L)$ has the following basis. (Show work).

   (c) Hence L is/not injective. (Choose the correct option. Explain.)

   (d) $Image(L)$ has the following basis. (Show work).

   (e) Hence L is/not surjective. (Choose the correct option.Explain.)

Q.3.

Let $B = (b_1 \ \ b_2 \ \ b_3)$ be a basis of a vector space V.
You are given three vectors in V:

$$c_1 = b_1 + 2b_2, \quad c_2 = b_1 + 3b_2 + b_3, \quad c_3 = 2b_1 + 5b_2 + b_3.$$

1. Define what is meant by a set of vectors $v_1, v_2, \cdots, v_r$ to be **linearly dependent**.

2. Give examples of three linearly dependent vectors in the given vector space V above. You must choose your three vectors from the six vectors $b_1, b_2, b_3, c_1, c_2, c_3$ and explain why they satisfy the definition.

3. Define what is meant by the coordinate vector of a given vector v with respect to a basis $(v_1 \ v_2 \ \cdots \ v_n)$.

4. Determine the coordinate vectors: $[c_1]_B, [c_2]_B, [c_3]_B$.

5. What is the $\dim(V)$?

6. Determine if $(c_1 \ c_2 \ c_3)$ is also a basis of V. Justify your answer. Calculator use is permitted, but you must explain the calculations.

Q.4.

Let M be an $n \times n$ matrix over real numbers. Complete the following definitions.

1. A vector $v \in \Re^n$ is an eigenvector for M if

____

2. A scalar λ is an eigenvalue for M if

____

3. A subspace W of $\Re^n$ is said to be an eigenspace of an eigenvalue λ for M if

____

4. Consider the matrix

$$M = \begin{pmatrix} 2 & 1 \\ -4 & 7 \end{pmatrix}.$$

Determine the characteristic equation and all the eigenvalues and corresponding eigenspaces of M.

Use your calculations to diagonalize M, i.e. to find a matrix P such that $M = PDP^{-1}$ where D is a diagonal matrix.

Answer. $P =$ ____________________ $D =$ ____________________

5. Calculate the characteristic equation and the eigenvalues for the following matrix.

$$H = \begin{pmatrix} 2 & 1 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Q.5.

Let $T: \ P_2 \to P_2$
be the linear transformation defined by the formula:
$T(p(x)) = p(x) - xp'(x)$ where $p'(x)$ is the first two derivatives of $p(x)$.
Let $B = (1 \ \ x \ \ x^2)$ be the standard basis of P_2.
Answer the following questions.

1. Calculate the images of all the basis vectors.

2. Determine the matrix M of the transformation T in the basis B.

3. Determine if the matrix M above is invertible. Show your work.

4. Calculate a basis for the $Ker\ (T)$ and decide if T is one to one.

5. Calculate a basis for the $Im\ (T)$ and decide if T is onto.

Q.6.

Recall the matrix A introduced at the beginning. Form the augmented matrix $H = (A|I_4)$. Answer the following questions.

1. Find an REF of H and call it H^*. Calculator use is allowed.

2. Consider the rows of H^* which have all zeros in the A part and let S the matrix formed by these rows in the last four columns.

   Calculate SA.

3. Explain why $Col\ A = Nul\ S$. Be sure to show that both sides have the same dimension and that one is contained in the other.

4. Explain why $S^T = (Col\ A)^{\perp}$. Be sure to show that both sides have the same dimension and that one is contained in the other.

Q.7.

Let $H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$.

1. Calculate the matrix $M = H^T H$ and its characteristic polynomial.

2. As usual,diagonalize the matrix M, i.e. write $M = PDP^{-1}$ where P is a matrix with eigenvectors of M as columns. **Choose columns of P in order of decreasing eigenvalues!**

3. Calculate HP and verify that its columns are two orthogonal vectors in $\Re^3$. Name the columns w_1, w_2 in order.

4. Find a third non zero vector w_3 such that w_3 is perpendicular to both w_1, w_2. This can be done in many different ways.

   Explain your method carefully.

5. Explain why w_1, w_2, w_3 must be a basis of $\Re^3$. You should state the correct theorem. No further calculations are needed.