DEPARTMENT OF MATHEMATICS

Ma322 - Final Exam Spring 2011
May 3,4, 2011

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Be sure to show all work and justify your answers.
There are 8 problems and a total of 9 pages including this one. No other sheets, books, papers are allowed.

Problem	Maximum Score	Actual Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
Free	4	
Total	100	

NAME: ______________________________

1. Let

$$A = \begin{pmatrix} 7 & -8 & 2 \\ 0 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 57 \\ -4 \\ 27 \end{pmatrix}$$

(a) Find the **adjoint** of the matrix A given above.

Note: You must first write the adjoint as a matrix of 2×2 minors with signs in correct position. Then you should give the evaluated expression. A calculator may be used for evaluation.

(b) Using the adjoint or directly, compute $\det(A)$.

(c) State the formula connecting the adjoint and the determinant of a matrix with its inverse.

Using it, determine the inverse A^{-1}.

(d) **Using the above inverse** solve the equation $AX = B$ for the given matrix B.

Note. No credit will be given if you use Gaussian elimination again to reduce $(A|B)$. Using the inverse is the main point!

Answer: $A^{-1} =$ $\qquad\qquad\qquad$ $X =$

2. You are given the augmented matrix of a linear system of equations.

   First reduce the system to REF using the standard algorithm.

   Then find the general solution to the system of equations.

   It is not necessary to find the RREF.

   Be sure to identify the operations in correct notation introduced in class, namely $kR_i, R_i + cR_j$ or P_{ij}.

$$\left[\begin{array}{ccccc|c} x & y & z & w & t & RHS \\ \hline 1 & 0 & 3 & 0 & 1 & -2 \\ 1 & 1 & 3 & 3 & -2 & 9 \\ 0 & -2 & 0 & -5 & 4 & -19 \end{array}\right]$$

3. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 & 5 \\ -2 & 3 & 5 & -6 & -8 \\ 2 & -3 & -5 & 4 & 12 \\ -4 & 6 & 10 & -12 & -16 \end{pmatrix} \text{ and } M = \begin{pmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 3 & 5 & 2 & 2 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

You are given that M is obtained from A by row transformations. Answer the following questions, using the matrix M as needed.

Answer the following questions.

(a) What is the dimension of $Col(A)$? Justify your claim by giving an explicit basis for $Col(A)$ and explaining why it is a basis.

(b) What is the dimension of $Nul(A)$. Justify your claim by giving an explicit basis for $Nul(A)$ and explaining why it is a basis.

(c) If L is the linear transformation $L : \Re^5 \to \Re^4$ given by $L(X) = AX$.

Answer the following:

i. State the Fundamental Theorem about the dimensions of various vector spaces associated with L.

ii. Use the above information to decide the dimensions of $Ker(L)$ and $Im(L)$.

iii. Verify the above mentioned Fundamental Theorem for the given transformation L.

4. You are given the following calculations:

$$A := \begin{pmatrix} 1 & -1 \\ 5 & 2 \\ 5 & -2 \\ 2 & 4 \end{pmatrix} \text{ and an REF of } (A|I) \text{ is: } M := \left(\begin{array}{cc|cccc} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -5 & 1 & 0 & 0 \\ 0 & 0 & -20 & -3 & 7 & 0 \\ 0 & 0 & 0 & -6 & 4 & 5 \end{array}\right)$$

(a) What is the rank of A? Why?

(b) What is **the "consistency matrix" G** for A as given by the above M ?

(c) Use the matrix G to find the value of x for which $\begin{pmatrix} 0 \\ 7 \\ 3 \\ x \end{pmatrix}$ is in $Col(A)$.

(d) Use the above calculations to write $Col(A) = Nul(B)$ for some matrix B. You must explain why your answer is correct.

5. (a) Consider the matrix

$$A = \begin{pmatrix} 2 & 9 \\ 2 & -5 \end{pmatrix}.$$

Determine the characteristic equation and all the eigenvalues and corresponding eigenspaces of A.

Use your calculations to diagonalize A, i.e. to find a matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

Answer. $P =$ ____________________ $D =$ ____________________

(b) Calculate the characteristic equation and the eigenvalues for the following matrix.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 9 \\ 0 & 2 & -5 \end{pmatrix}.$$

(c) Is the matrix M diagonalizable? You must state the appropriate theorem and explain why it applies.

6. Let $T: \ P_2 \to P_2$

   be the linear transformation defined by the formula:

$$T(f(x)) = 2\,x^2 \frac{d^2}{dx^2} f\,(x) - 4\,f\,(x)\,.$$

   Let $B = (1 \quad x \quad x^2)$ be the standard basis of P_2.

   Answer the following:

   (a) Let $L : V \to V$ be a linear transformation.
   Define the following:

      i. A real number λ is said to be an eigenvalue for L if:

      ii. If λ is an eigenvalue for L, then its eigenspace V_λ is defined as:

   (b) Calculate the images by T of all the basis vectors in B.

   (c) Determine the matrix M of the transformation T in the basis B.

   (d) Determine all the eigenvalues of T and the corresponding eigenspaces.
   Hint: These can be calculated by inspection alone. Be sure to write down your reasoning.

7. An inner product matrix for a 3-dimensional space with basis $B = (v_1\ v_2\ v_3)$ is given to be

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 1 \\ 2 & 1 & 15 \end{pmatrix}$$

Hint: Recall that this means $< v_i, v_j >= M(i, j)$.

Answer the following questions.

(a) Use the algorithm explained in the class notes on Inner Products to make an orthogonal basis for V starting with B.

Hint: Start with $(M|I)$. The algorithm produces an upper triangular matrix R such that $R^T MR$ is a diagonal matrix. The desired basis is then BR.

(b) Use the inner product matrix M to determine the following quantities:

i. $||v_1||$, $||v_2||$.

ii. $< v_1, v_2 - 2v_1 >$.

iii. The angle between v_1 and $v_2 - 2v_1$.

8. Define an inner product on the vector space P of all polynomials by the formula

$$< p(x), q(x) >= \int_{-1}^{1} p(t)q(t)dt.$$

$$\text{Let } f(x) = 1 + 2\,x, \quad g(x) = 1 - 3\,x^2, \quad h(x) = 3\,x - 2.$$

Answer the following questions based on this inner product.

(a) Calculate the pairwise inner products. Be sure to show your work.

$$< f(x), g(x) >, < g(x), h(x) >, < f(x), h(x) > .$$

(b) Using the above information or otherwise, explain why $B = (f(x)\ g(x)\ h(x))$ is an orthogonal basis of P_2. Be sure to state any necessary theorems.

(c) Calculate $[x]_B$, the coordinate vector of the polynomial x in the basis B. Be sure to use the orthogonality of B. Show all work.

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Key

1. Adjoint=inverse $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -3 & 4 & 7 \end{bmatrix}$

   Answer: $X = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$.

2. REF $\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & -2 \\ 0 & 1 & 0 & 3 & -3 & 11 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{bmatrix}$

   RREF $\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{bmatrix}$

3. (a) 3 since M has pivot rank 3. Basis columns are $1, 2, 4$.

   (b) 2 since $colnum(A) - rank(A) = 2$.

   RREF is $\begin{bmatrix} 1 & 0 & 0 & 0 & 13 \\ 0 & 1 & 5/3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   (c) $\dim(\Re^5) = \dim(Ker(L)) + \dim(Im(L))$.

   (d) Kernel 2, Image 3.

   (e) $5 = 2 + 3$.

4. (a) 2 since we see 2 pivots in REF.

   (b) $G = \begin{bmatrix} -20 & -3 & 7 & 0 \\ 0 & -6 & 4 & 5 \end{bmatrix}$.

   (c) Gv is $\begin{pmatrix} 0 \\ -30 + 5x \end{pmatrix}$. Hence $x = 6$.

   (d) $B = G$.

5. (a) $(t + 7)(t - 4)$.

   (b) For $\lambda = -7$: $\begin{bmatrix} 9 & 9 \\ 2 & 2 \end{bmatrix}$

   For $\lambda = 4$: $\begin{bmatrix} -2 & 9 \\ 2 & -9 \end{bmatrix}$.

   $P = \begin{pmatrix} 1 & 9 \\ -1 & 2 \end{pmatrix}, D = \begin{pmatrix} -7 & 0 \\ 0 & 4 \end{pmatrix}$.

(c)
$$(1-t)(t+7)(t-4)$$
.

$1, -7, 4.$

For 1, use $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. For others lift the above adding 0 on top.

(d) Yes, because it has distinct e-values.

6. (a) defs.

(b) The matrix is $\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So, the image has basis $1, x$.

(c) Matrix found above.

(d) E-values $-4, 0$. For -4 the e-space is e_1, e_2. For 0 it is e_3.

7. (a) Resulting matrix
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & -8 & 3 & 1 \end{bmatrix}$$
.

(b) $1, 1.$

(c) $2 - 2 = 0.$

(d) $\pi/2.$

8. (a) All 0.

(b) Orthogonal and 3 in number, so basis.

(c) $\begin{pmatrix} 2/7 \\ 0 \\ 1/7 \end{pmatrix}$.