DEPARTMENT OF MATHEMATICS

Ma322 - FINAL EXAM Spring 2012
May 3, 2012

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Be sure to show all work and justify your answers.
There are 8 problems and a total of 9 pages including this one. No other sheets, books, papers are allowed.

Problem	Maximum Score	Actual Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
Free	4	
Total	100	

NAME: ______________________________

SECTION NO: ______________________________

1. **Basic Definitions.**

(a) Define what is meant by **the rank of a matrix,** by completing the following sentence.

Definition. An $m \times n$ matrix H is said to have rank d if $\cdots$

(b) Define what is meant by **a linear system of equations to be consistent,** by completing the following sentence.

Definition. A linear system of equations $HX = Y$ is said to be consistent if $\cdots$

(c) Let

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 0 \\ -1 & -2 & 1 & 1 & 1 \\ 0 & 3 & -9 & -12 & -3 \end{pmatrix}.$$

Now illustrate the above definitions using A as follows. You **must use your definitions in each case!**

- **Calculate the rank of A.**

- **Find a vector B such that $AX = B$ is inconsistent. Justify your answer.**

- **Find another vector C such that $AX = C$ is consistent. Justify your answer.**

2. **More definitions.**

(a) Define what is meant by a linear transformation by completing the following:
Definition. Let V, W be vector spaces over $\Re$.
A map $L : V \to W$ is said to be a linear transformation if $\cdots$

(b) Let $L : V \to W$ be a linear transformation of real vector spaces. Define the following:

- $Ker(L) = \left\{ v \in V \;\middle|\; \qquad\qquad\qquad\qquad \right\}.$

- $Im(L) = \left\{ w \in W \;\middle|\; \qquad\qquad\qquad\qquad \right\}.$

(c) Define a linear transformation $L : \Re^n \to \Re^m$ by the formula $L(X) = AX$ where
$A = \begin{pmatrix} 1 & 1 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$
Answer the following questions.

i. because of the size of A, we must have $n =$ _____ and $m =$ _____

ii. $Ker(L)$ has the following basis. (Show work).

iii. Hence L is/not injective. (Choose the correct option. Explain.)

iv. $Image(L)$ has the following basis. (Show work).

v. Hence L is/not surjective. (Choose the correct option. Explain.)

3. **Abstract vector space.**

Let $B = (b_1 \;\; b_2 \;\; b_3)$ be a basis of a vector space V.

You are given three vectors in V:

$$c_1 = b_1 + 2b_2, \quad c_2 = -b_1 + 3b_2 + 4b_3, \quad c_3 = 5b_1 - b_2 + 6b_3.$$

(a) Define what is meant by a set of vectors $v_1, v_2, \cdots, v_r$ to be **linearly dependent**.

(b) Give an example of three **linearly dependent vectors chosen from the six vectors** $\{b_1, b_2, b_3, c_1, c_2, c_3\}$.
You **must explain** why they satisfy the definition.

(c) Define what is meant by the coordinate vector of a given vector v with respect to a basis $(v_1 \; v_2 \; \cdots \; v_n)$.

(d) Determine the coordinate vectors: $[c_1]_B, [c_2]_B, [c_3]_B$.

(e) Let $W = Span\{b_1, c_1, c_2\}$. What is the dimension of W?
Justify your answer.
You must show calculations. Calculator declaration is not accepted!

4. **Eigenspaces.**

Let M be an $n \times n$ matrix over real numbers. Complete the following definitions.

(a) A vector $v \in \Re^n$ is an eigenvector for M if

____

(b) A scalar λ is an eigenvalue for M if

____

(c) A subspace W of $\Re^n$ is said to be an eigenspace of an eigenvalue λ for M if

____

(d) Consider the matrix

$$M = \begin{pmatrix} -2 & 1 \\ -4 & -7 \end{pmatrix}.$$

Determine the characteristic equation and all the eigenvalues and corresponding eigenspaces of M.

Use your calculations to diagonalize M, i.e. to find a matrix P such that $M = PDP^{-1}$ where D is a diagonal matrix.

Answer. $P =$ ________________ $D =$ ________________

(e) Calculate the characteristic equation and the eigenvalues for the following matrix.

$$H = \begin{pmatrix} -2 & 1 & 0 \\ -4 & -7 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. **Linear Transformations.**

Let $T: \ P_3 \to \Re^2$

be the linear transformation defined by the formula:

$T(p(x)) = \begin{pmatrix} p(0) \\ p(2) \end{pmatrix}$. Thus, $T(1 - x + 3x^3) = \begin{pmatrix} 1-0+0 \\ 1-2+24 \end{pmatrix} = \begin{pmatrix} 1 \\ 23 \end{pmatrix}$.

Let $B = (1 \ \ x \ \ x^2 \ \ x^3)$ be the standard basis of P_3.

Answer the following questions.

(a) Calculate the images of all the basis vectors.

$$T(1) = \qquad , T(x) = \qquad , T(x^2) = \qquad , T(x^3) = \qquad .$$

(b) Determine the matrix M of the transformation T using the basis B for P_3 and the usual $E = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ for $\Re^2$.

Hint: The matrix will be 2×4.

(c) Calculate a basis for the $Ker\ (T)$.

Hint: The answer must be a sequence of vectors in P_3.

(d) Calculate a basis for the $Im(T)$.

Hint: The answer must be a sequence of vectors in $\Re^2$.

(e) Is the transformation T injective (one to one)? You must explain, just a yes or no would not earn credit.

(f) Is the transformation T surjective (onto)? You must explain, just a yes or no would not earn credit.

6. **The Consistency Matrix.**

You are given:

$$A = \begin{pmatrix} 1 & -1 \\ -2 & -4 \\ -2 & 5 \\ -4 & -1 \end{pmatrix} \text{ and an REF of } (A|I) \text{ is: } M = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -6 & 2 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 2 & 0 \\ 0 & 0 & 0 & -11 & -7 & 9 \end{pmatrix}.$$

Let H denote the matrix formed by the last four columns of M. Answer the following questions using these calculations.

(a) What is the consistency matrix G of A as obtained from the above information?

(b) Use the consistency matrix G to determine which of these vectors are in $Col(A)$.

$$v = \begin{pmatrix} -3 \\ 18 \\ 0 \\ 22 \end{pmatrix}, w = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix}.$$

Important. It is important to show your calculations. Just a yes or no would not earn any points.

(c) Explain why $Col(A) = Nul(G)$. Be sure to show that both sides have the same dimension and that one is contained in the other.

7. **Inner Products in abstract spaces.**

Consider the vector space P of all polynomials in one variable x with real coefficients. For this question, define the inner product on P by the formula

$$< p(x), q(x) >= \int_0^1 p(t)q(t)\,dt.$$

Answer the following questions.

(a) Calculate the following inner products.
Hint: Make a general formula for $< x^i, x^j >$ and apply it.

$$< 1, 1 >, < 1, x >, < 1, x^2 >, < x, x^2 > .$$

(b) Calculate the angle between the vectors x and x^2. You may use the calculator, but must write down the different parts of the formula separately.

(c) Calculate a value of a for which 1 and $(x - a)$ are mutually perpendicular. Show all work.

(d) State the Cauchy-Schwartz inequality. Be sure to also state the condition when the inequality becomes an equality.

(e) Find the projection of x^3 on $Span\{x^2\}$.

8. **Inner product and projection in $\Re^n$.**

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -2 & -6 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}.$$

For this question, use the usual inner product (dot product) in $\Re^4$.

Answer the following questions.

(a) Start with the equation system $AX = B$. Explain why the system is not consistent. The aim of the following questions is to find a best approximation to the solution of this system.

(b) Write down the normal equations derived from the above.

(c) Solve the normal system. Show all work. You should find the inverse of the 2×2 matrix by the formula for the inverse during your work.

(d) Use your solution to find the projection of B onto the $Col(A)$. (The answer must be a vector in $\Re^4$).