**NAME:** __________________ **Section:** ______ GUID ______

# FINAL EXAM

Ma322-003 (Eakin) Fall 2015

December 16, 2015

**Please make sure that your name and GUID are on every page.**

**This exam is designed to be done with pencil-and-paper calculations. You may use your calculator for routine arithmetic. You must make it clear how you arrived at your answers. Answers simply taken from calculators will receive no credit.**
There are 12 problems and a total of 12 pages including this one. No other sheets, books, papers are allowed.
**All calculated answers must be justified!!**. If you need extra space please use the backs of the exam pages. If you do so then be sure to direct the scorer to the location of the additional work and to make it clear which additional information goes with which problem. **Everything submitted will be graded.** Be sure to erase or cross out any work or text that is to be ignored.

| **Problem** | **Maximum Score** | **Actual Score** |
|---|---|---|
| 1 | 15 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 8 | |
| 6 | 8 | |
| 7 | 8 | |
| 8 | 10 | |
| 9 | 12 | |
| 10 | 8 | |
| 11 | 8 | |
| 12 | 8 | |
| Total | 100 | |

NAME: ____________ **Section:** ______ GUID ______

1. (15 points) Complete each of the following to provide proper definitions or complete, general descriptions. Operational definitions (i.e. descriptions of how the object is calculated ) will not suffice.

   (a) If $\Re$ denotes the real numbers then precisely $\Re^3$ is defined to be:

   (b) If $S = \{v_1, v_2, v_3, v_4\}$ is a set of vectors in $\Re^n$ then

      i. $S$ is **linearly dependent** if

      ii. If $S = \{v_1, v_2, v_3, v_4\}$ is a set of vectors in $\Re^n$ then $S$ is **linearly independent** if

      iii. the **linear span** of $S$ is

   (c) $V \subset \Re^n$ is a **subspace** of $\Re^n$ if

   (d) $B = \{b_1, \cdots, b_m\}$ is a **basis** for the vector space $V \subset \Re^n$ if

   (e) If $V$ is a vector space then the **dimension** of $V$ is

(f) If $A$ is an $m$ by $n$ matrix then the **column space** of $A$ is

(g) If $A$ is a matrix then the **rank** of $A$ is the dimension of

(h) If $A$ is an $m$ by $n$ matrix then the **null space** of $A$ is

(i) If $A$ is an $m$ by $n$ matrix and $B \in \Re^m$ then the linear system $AX = B$ is **consistent** if

(j) If $A$ is an $n$ by $n$ matrix then $A$ is **invertible** if

(k) If $A$ is an $n$ by $n$ matrix then $\alpha$ is an **eigenvalue** of $A$ if

(l) If $A$ is an $n$ by $n$ matrix and $v \in \Re^n$ then $v$ is an **eigenvector** of $A$ for the eigenvalue $\alpha$ if

(m) If $A$ is an $n$ by $n$ matrix then the **characteristic polynomial** of $A$ is

(n) If $V \subset \Re^n$ is a vector space then the **orthogonal complement** of $V$ is

NAME: _____________ **Section:** ______ GUID ______

(o) If $v \in \Re^n$ is a non-zero vector then the **unit vector** having the same direction as $v$ is

(p) If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors in $\Re^3$ then

$< X, Y >=$ ____________________ and

$||X||$, the **length** of $X$ is _____________

(q) If $T$ is a mapping from $\Re^n$ to $\Re^m$ then $T$ is a **linear transformation** if:

(r) If $T$ is a linear transformation from $\Re^n$ to $\Re^m$ then
i. $T$ is **onto (surjective)** if

ii. $T$ is **one to one (injective)** if

NAME: ______________ **Section:** _______ GUID _______

2. (5 points) Let $A$ be a square matrix with columns $A_1, A_2, A_3$.

   (a) Explain why determinant($A$) =0 if $A_1 = A_3$

   (b) Explain why determinant($A$) =0 if $A_3 = 2A_1 - A_2$

3. (5 points) Calculate the determinant of $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix}$ by the Laplace (cofactor) expansion down the second column. You can leave your answer in the form of a sum and you do not need to calculate the smaller determinants. (Show your work.)

4. (5 points) Suppose that $A$ and $B$ are subspaces of $\Re^7$. If $A$ is 3 dimensional and $B$ is 5 dimensional, show that there must be a non-zero vector that is in both $A$ and $B$.

NAME: ____________ **Section:** ______ GUID ______

5. (8 points) Suppose $\{A, B, C\}$ is a linearly independent set of elements of the vector space $V$.

   (a) Show that $\{A, B, A + B + C\}$ is linearly independent.

   (b) Show that $\{A, A + B, A + 2B\}$ is linearly dependent.

6. (8 points)Suppose $T : \Re^3 \to \Re^4$ is a linear transformation and that $\{A, B, C\}$ is a linearly independent subset of $\Re^3$.

   (a) If $T$ is injective (one-to-one) show that $\{T(A), T(B), T(C)\}$ is a linearly independent subset of $\Re^4$.

   (b) If $T(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) = T(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = B$

      a Give an explicit, non-zero element of the kernel of $T$.

      b Find an infinite number of different, non-zero elements $X$ such that $T(X) = B$.

7. (8 points) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$ and $V$ its column space. Matrices related to $A$ can be found in the supplemental pages.

(a) Calculate $P_V$, the matrix for the orthogonal projection of $\Re^3$ onto $V$. Express the answer as a product of specific matrices. Do not bother to multiply the matrices.

**ANS:**
$A(A^tA)^{-1}A^t$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

(b) Do the following calculations. They can very easily be done by hand. You may use a calculator but must provide sufficient detail to make it clear that your answer isn't simply coming from a calculator routine.

Let $Y = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$.

i. Calculate $P_V(Y)$.

**ANS:** Most will have done it by calculator. However I think the instructions are clear that some detail of the calculation has to be there. This has form (RST)X - something like RS=W, WT= Q, QX = ans would be fine. In view of the instructions an unsupported answer gets nothing.

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

item Calculate the orthogonal projection of $Y$ into $V^{\perp}$.

$$Y = Y_V + Y_{V^\perp}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + Y_{V^\perp}, Y_{V^\perp} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

Of course some will calculate $B = I - P_V$ and calculate $BY$ with the calculator- that is ok as long as there is sufficient detail.

8. (10 points) If $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ then

(a) Calculate the eigenvalues of $A$

**ANS:** $det(A - xI) = det(\begin{pmatrix} 1-x & 2 \\ 0 & 3-x \end{pmatrix} = (1-x)(3-x)$ has roots $\lambda = 1, 3$.

(b) For each eigenvalue of $A$, calculate a basis for the corresponding eigenspace.

**ANS:**

(c) Find an invertible matrix $S$ such that $S^{-1}AS = D$ where $D$ is a diagonal matrix.

$\lambda = 1$ $\begin{pmatrix} 1-1 & 2 \\ 0 & 3-1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$

by inspection $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector. The dimension of the nullspace is 1 so this is a basis.

$\lambda = 3$ $\begin{pmatrix} 1-3 & 2 \\ 0 & 3-3 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$

by inspection $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector. The dimension of the nullspace is 1 so this is a basis.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(d) Verify that $S^{-1}AS = D$.

**ANS:**

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

9. (12 points) Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$. See the supplemental pages for matrices related to $A$.

   (a) Apply the Gram-Schmidt process to find a matrix $S$ such that $AS$ has orthogonal columns. (You are not asked to produce orthonormal columns, nor do you need to actually multiply $A$ times your answer).

   **Ans:** If $R =< U|T >$ is an REF of $< A^tA|I >$ with $U$ upper triangular then $S = T^t$. The supplemental handout for the exam contained an REF for this $A$ (item 1) from which one can read
   $S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
   I unfortunately had used $S$ for what I call $T$ in class so students may be be confused and use the matrix $T$ rather than $T^t$. If that happens deduct just 1 point.

   (b) Calculate the QR factorization of $A$. That is, find an orthogonal matrix $Q$ and an upper triangular matrix $R$ such that $A = QR$. You can express each of $Q$ and $R$ as a product of specific matrices. You do not need to do the matrix multiplication. Note that "orthogonal matrix" means that the columns are orthonormal.

   **Ans:** $Q$ is just the matrix $RS$ from $A$, normalized to make the columns unit vectors.
   $Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$
   The $R$ in $QR$ is just that diagonal times $U$
   $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}$

   (c) If $V$ is the column space of $A$, calculate a basis for the orthogonal complement of $V$.

   **ANS:** The columns of the consistency matrix of $A$ from any REF of $< A|I >$ is a basis. One is given in item(1) from which one reads $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is a basis.

   (d) Show that your basis from part (c) is orthogonal to each column of $A$.

   **ANS:** $< \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} >=< \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} >= 0$

10. (8 points) $V$ is the column space of the matrix $A = \begin{pmatrix} 1 & 0 & -1 & 5 & 2 \\ -1 & 1 & 1 & -1 & -3 \\ -1 & -1 & 1 & -9 & -1 \\ 1 & 1 & 1 & 5 & -1 \end{pmatrix}$

and the RREF of $< A|I >$ is $R = \begin{pmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 4 & -1 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(a) Determine a basis, $B$, for $V$.

**ANS:** The pivot cols of $A$, $\{A_1, A_2, A_3\}$ are the way we did this.

(b) What is the dimension of $V$? Explain your answer.

(c) Explain why $V$ is or is not all of $\Re^4$.

(d) If $A_4$ is column 4 of $A$, what is $[A_4]_B$, the coordinate vector for $A_4$ relative to your basis from part $(a)$?

**ANS:** I expect some will write $A_4 = 3A_1 + 4A_2 - 2A_3$ and not write $[A_4]_B = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$. I would go ahead and give them credit in that case.

11. (8 points) Let $data = \begin{array}{|c|c|} x & y \\ \hline -1 & 1 \\ 0 & 0 \\ 1 & 2 \end{array}$.

Calculate the least squares line $y = mx + b$ which best fits the given data. See the supplemental pages for matrices related to this problem. For this problem you are asked to calculate the $m$ and $b$.

**ANS:** If $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, then then $\begin{pmatrix} m \\ b \end{pmatrix} = (A^tA)^{-1}A^t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

$y = \frac{1}{2}x + b$

12. (8 points) Let $E = [e_1, e_2]$ be the standard basis for $\Re^2$ and $C = [c_1, c_2] = [2\,e_1 + 3\,e_2, 7\,e_1 + 10\,e_2]$

(a) Find a matrix $Q$ such that $C = EQ$

$Q = \begin{pmatrix} 2 & 7 \\ 3 & 10 \end{pmatrix}$

(b) Let $[X]_E$ denote the coordinate vector of $X$ relative to the standard basis, $E$. If $Y$ is the vector such that $[Y]_E = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, calculate $[Y]_C$, the coordinate vector of $Y$ relative to the basis $C$.

**ANS:** $E = Q^{-1}C$, $Y = E[Y]_E = CQ^{-1}[Y]_E$

$[Y]_C = Q^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -10 & 7 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

If you have work on this page which you intend to be graded, be sure to note the problem(s) to which it refers and to make a note on the problem to look here. Be careful to mark out or erase any work that is not to be graded.

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# Matrices Which May be Useful

The matrices below are all called "$A$" to match the usage in the problems. Be careful that the one(s) you use correspond to the particular problem you are solving. The numbers do not correspond to the problem numbers, nor do the matrices necessarily occur in the same order as in the problems. Not all of these matrices are relevant to problems on this exam.

1. $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$

   $A^tA = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

   $AA^t = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{pmatrix}$

   $\text{REF} < A^tA|I >= \begin{pmatrix} 2 & 2 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{pmatrix}$

   $\text{REF} < AA^t|I >= \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$

   $(A^tA)^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix}$

   $\text{REF} < A|I >= \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}$

2. $A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 2 \end{pmatrix}$

   $\text{REF} < A^tA|I >= \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & \frac{9}{2} & \frac{-1}{2} & 1 \end{pmatrix}$

   $\text{REF} < A|I >= \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

   $(A^tA)^{-1} = \begin{pmatrix} \frac{5}{9} & \frac{-1}{9} \\ \frac{-1}{9} & \frac{2}{9} \end{pmatrix}$

3. $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$\text{REF} < A^tA|I >= \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$$

$$\text{REF} < AA^t|I >= \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

$$\text{REF} < A|I >= \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$(A^tA)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

4. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$

$$\text{REF} < A^tA|I >= \begin{pmatrix} 5 & 3 & 1 & 0 \\ 0 & \frac{6}{5} & \frac{-3}{5} & 1 \end{pmatrix}$$

$$\text{REF} < A|I >= \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$(A^tA)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{6} \end{pmatrix}$$

5. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$

$$\text{REF} < A^tA|I >= \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}$$

$$\text{REF} < AA^t|I >= \begin{pmatrix} 5 & 2 & -3 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{-2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\text{REF} < A|I >= \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$(A^tA)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix}$$