1 Terminology

1. **Def.1: A Linear Function** of variables $x_1, \cdots, x_n$ is an expression of the form $a_1x_1 + \cdots + a_nx_n - b$ where $a_1, a_2, \cdots, a_n, b$ are assumed to be numbers (or scalers) in a field F.
2. Moreover **Def.2: A Linear Equation** of variables $x_1, \cdots, x_n$ is an equation prearranged as $a_1x_1 + \cdots + a_nx_n = b$. A set of linear equations in the same variables is called a **System of Linear equations.**
3. In practice, the coefficients may be allowed to have some parameters or temporary variables to be replaced by convenient scalars later on.
4. **Def.3: Augmented Row** To each linear equation as above we associate an augmented row with $n + 1$ entries

   $$\begin{pmatrix} a_1 & a_2 & \cdots & a_n & b \end{pmatrix} \text{ or for clarification of RHS } \begin{pmatrix} a_1 & a_2 & \cdots & a_n & | & b \end{pmatrix}$$
5. **Def.4: Augmented Matrix** To a system of Linear equations (i.e. a set of linear equations) we associate an augmented matrix by stacking their augmented rows one above the other.
6. We also add a title row of variables for convenience.
7. For example, a system of linear equations

   $$x + y + 3z = 12, y + 6z = 20, 2x + 3z = 11, -x + y - 2z = -5$$

   gives an augmented matrix:

   $$\left(\begin{array}{rrr|r} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 2 & 0 & 3 & 11 \\ -1 & 1 & -2 & -5 \end{array}\right)$$

2 Basics of the solution process.

1. **Def. 5. A solution of a system of linear equations.** Given one equation $f(x_1, \cdots, x_n) = b$, an n-tuple $(s_1, \cdots, s_n)$ is said to be its solution if $f(s_1, \cdots, s_n) = b$.

   We shall shorten the above notation for convenience by writing the equation as $f(X) = b$ and the solution as $s = (s_1, \cdots, s_n)$, so that we can say s is a solution of $f(X) = b$ if $f(s) = b$.

   If we have a set of equations $E = \{f_1(X) = b_1, \cdots, f_m(X) = b_m\}$, then we say that s is a solution of the system E if $f_i(X) = b_i$ for each i.
2. **Def. 6. Equivalent systems of linear equations.** Two systems of linear equations E, E^* are defined (in the book) as systems having the same set of solutions. Even though this is correct, it is not too useful since it involves solving both and verifying that the solutions match.

   We use a better test: Two systems E, E^* are equivalent if each equation in E is a linear combination of equations in E^* and conversely, each equation in E^* is a linear combination of equations in E.
3. We note that shuffling the rows of the augmented matrix describes the same system of equations. In general, we record the following elementary row operations on our matrix which are easily shown to give equivalent linear systems.
4. **Def. 7.: Elementary Row Operations.**
   (a) Swap an i-th row with a different j-th row. Notation: P_{ij}.
   (b) Multiply an i-th row by a non zero scalar k Notation: kR_i.
   (c) Multiply the j-th row by a scalar c and add it to a different i-th row. Notation: $R_i + cR_j$.
5. These notations must be learnt and memorized precisely!

3 A sample of elementary operations.

1. We now perform elementary row operations on the above augmented matrix to illustrate how the equations are simplified and hence solved.

$$\left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 2 & 0 & 3 & 11 \\ -1 & 1 & -2 & -5 \end{array}\right) \begin{pmatrix} R_3 - 2R_1 \\ R_4 + R_1 \end{pmatrix} \left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & -2 & -3 & -13 \\ 0 & 2 & 1 & 7 \end{array}\right)$$

2.
$$\left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & -2 & -3 & -13 \\ 0 & 2 & 1 & 7 \end{array}\right) \begin{pmatrix} R_3 + 2R_2 \\ R_4 - 2R_2 \end{pmatrix} \left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 9 & 27 \\ 0 & 0 & -11 & -33 \end{array}\right)$$

3.
$$\left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 9 & 27 \\ 0 & 0 & -11 & -33 \end{array}\right) R_4 - \frac{-11}{9} R_3 \left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 9 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

4.
   - If we read the current equations from bottom to top, we see:
   $$0 = 0, \quad 9z = 27, \quad y + 6z = 20, \quad x + y + 3z = 12.$$
   - These can be solved in order for z, y, x respectively, to deduce
   $$z = 3, y = 2, x = 1.$$
   - Recall that our equations from bottom to top, were:
   $$0 = 0, \quad 9z = 27, \quad y + 6z = 20, \quad x + y + 3z = 12.$$
   - We can ignore $0 = 0$, since it is always true!
   - Thus, $9z = 27$ gives $z = 3$.
   - Now $y + 6z = 20$ gives $y + 18 = 20$ or $y = 2$.
   - Now $x + y + 3z = 12$ gives $x + 2 + 9 = 20$ or $x = 1$.

5. **The $\mathbf{0}, 1, \infty$ principle:**
   - Sometimes, our system does not lead to any solution. For instance, a system $2x + 3y = 5, 4x + 6y = 6$ would leads to:
   $$\left(\begin{array}{cc|c} x & y & RHS \\ \hline 2 & 3 & 5 \\ 4 & 6 & 6 \end{array}\right) R_2 - 2R_1 \left(\begin{array}{cc|c} x & y & RHS \\ \hline 2 & 3 & 5 \\ 0 & 0 & -4 \end{array}\right).$$
   - The second equation is $0 = -4$ and such equations are said to be **inconsistent.** The equations have 0 solutions (i.e. no solutions).
   - If we change the above equations to $2x + 3y = 5, 4x + 6y = 10$, then it is easily seen that the system has an infinity of solutions, namely $x = 1 - 3t, y = 1 + 2t$ where t can take any value!
   - We will later show that any linear system of equations has 0, 1 or infinitely many solutions!

6. **Understanding the process:**
   - Thus, our work is M_1 to M_2 to M_3, where:

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$$M_1 = \left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 2 & 0 & 3 & 11 \\ -1 & 1 & -2 & -5 \end{array}\right) \to M_2 = \left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 9 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  and

$$M_3 = \left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

7. **The PC list**
   - If a row has at least one non zero entry, **"the pivot of that row "** shall be the first nonzero entry in it.
   - The book calls this as the "leading entry" and the corresponding column, the pivot column.
   - **Def.8: pivot column (pc) number of a row** is defined to be the column number of the pivot of that row.
   - Thus, the pc number for $\begin{pmatrix} 0 & 0 & 2 & 0 & -3 \end{pmatrix}$ is 3 whereas it is ∞ for $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
   - For a row full of zeros, there is no pivot entry and the "pc number " is defined to be ∞.
   - **Def.9: pivot column (pc) list of a matrix** is the list of the pc numbers of successive rows of that matrix.

8. **More definitions:**
   - Thus, we see that the pc lists of the following matrices are respectively $(1, 2, 2), (1, 1, 3), (2, \infty, 1)$.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 2 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 7 \end{pmatrix}$$

   - We say that **Def.10: The pc list is strict** if it is an increasing sequence. Note that for this definition, we accept that a sequence of ∞ as increasing.
   - For our matrices M_1, M_2 the pc lists were: $(1, 2, 1, 1)$ and $(1, 2, 3, \infty)$. For M_3 it stays the same as M_2.
   - **Def.11: A matrix is said to be in REF if its pc list is strict.**
   - Thus we accept $(1, 3, \infty, \infty)$ as a strict sequence also!