# 1 Topics

1. Review of the Standard Gauss Elimination Algorithm: REF+ Backsub
2. The rank of a matrix.
3. Vectors and Linear combinations.
4. Span of a set of vectors.
5. Linear equations as a test of membership to a span.
6. The system of an augmented matrix $(A|B)$ reinterpreted as $AX = B$.

# 2 Gaussian Elimination.

1. By a linear system of equations we mean an augmented matrix $(A|B)$ with or without a title row specified. The separator bar is also optional.

   As we have seen, it is useful to have the system reduced to REF (i.e. with a strict pc list).
2. The big theorem is that a suitable sequence of elementary operations applied to this matrix will always produce such an REF. There is no claim (or hope) of uniqueness of the final form, however, a certain associated integer called its rank will turn out to be a very useful tool for the solution process.
3. While there are many choices of operations, we describe a certain well defined choice which is guaranteed to work. This will be described as the standard algorithm.

# 3 The standard algorithm.

1. Here is the set up. We assume that our augmented matrix is renamed $M$ and it has $m$ rows. We assume that
   - The top $i$ rows of $M$ are already in REF and
   - All the pivot columns in rows $i + 1$ to $m$ are strictly bigger than the pivot columns for the first $i$ rows.
2. We describe this situation as having the top $i$ rows inactive.
3. The Plan: We can always begin with $i = 0$ and our algorithm will push $i$ to $m$. Clearly, at the end, we have REF!
4. Now we show how to make the $i + 1$-th row inactive. This is the iterative step.
5. Among the rows from $i + 1$ to $m$, we pick the one whose pivot column number is the least, say $s$.

   We do a row swap **only if needed** to make this row the $i + 1$-th.
6. Important rule. We always choose the smallest numbered row to swap into this $(i + 1)$-th place. This is the only time we use a row swap, and only if really needed.
7. Now we do a sequence of row operations to arrange the pivot column numbers of all rows from $i + 2$ to $m$ to be bigger than the pc $s$ of the $(i + 1)^{th}$ row.
   - For each $j > i + 1$ we do the following well defined operation.
   - Consider the pivot entry of the $(i + 1)^{th}$ row, namely $M(i + 1, s)$.
   - For each $j > i + 1$ we wish to arrange the pc of the $j$-th row to be bigger than $s$. We already know that it is at least $s$.
   - This exactly means that we need $M(j, s) = 0$ for all such $j > i + 1$. We call this entry $M(j, s)$ the target - to be made zero!

- **Important Formula:** We use the operation $R_j - cR_{i+1}$ where $c$ is given by the formula $c = \frac{M(j,s)}{M(i+1,s)}$.
- Note that the formula for $c$ can be remembered as $\frac{target}{pivot}$.
- Note that this step is carried out for each $j > i+1$ whenever $M(j,s) \neq 0$. We typically do it in sequence, but as long as $i+1$ is fixed, all these steps can be done at the same time, since they do not interfere with each other!

# 4 Using the algorithm.

1. As seen above, we can make all the $m$ rows inactive and thus have REF. The pc-list is now strict and thus all rows which become zero appear only after the non zero rows.
   - At this stage, we are ready to solve the original equations.
   - **Def.12: Rank of $M$.** The number of pivots in the final REF is called the rank of $M$ and is denoted by $rank(M)$.
   - Note that we have not proved the rank to be well defined. That proof will come much later.
   - Write the final form as $M^* = (A^*|B^*)$.
   - We note that both $A^*$ and $M^*$ are in REF and these are respectively REF of $A$ and $M$.
2. **Def.13: Consistency:** A system $(A|B)$ is said to be consistent if it has at least one solution.
3. **Def.14: Consistency Condition.** We note that the original system represented by $(A|B)$ is consistent if and only if $rank(A) = rank(M)$.
4. Explicitly, this means that all the pivots in $M^*$ occur in the $A^*$ part. In other words, if some row of $A^*$ is zero, then it must be also the zero row of $M^*$.
5. Example of an inconsistent system.
   - Consider our old example with the RHS changed in the last equation.

$$\begin{pmatrix} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 2 & 0 & 3 & 11 \\ -1 & 1 & -2 & t \end{pmatrix}$$

   - It can be shown that the same REF steps as before, produce:

$$\begin{pmatrix} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 2 & 0 & 3 & 11 \\ -1 & 1 & -2 & t \end{pmatrix} \to \begin{pmatrix} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 9 & 27 \\ 0 & 0 & 0 & t+5 \end{pmatrix}$$

   - Thus, our original system is consistent if and only if $t+5=0$ or $t=-5$.
   - Our original system had $t=-5$ and hence was consistent! For that system, both $A$ and $M$ had the same rank 3.

# 5 Vector Spaces.

1. Now we present a different way of understanding our work.

   **Def.15: Vectors in $n$-dimensions.** A column of $n$ scalars $v = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{pmatrix}$ is said to be an $n$-dimensional vector.

2. **Def.16: The set of all $n$-dimensional vectors forms the vector space $\Re^n$.** The space $\Re^n$ has two natural operations.

   - Given $v = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{pmatrix}$ and $w = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$ we define addition

$$v + w = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \cdots \\ a_n + b_n \end{pmatrix}.$$

   - Further, for any given scalar $c$, we define scalar multiplication

$$cv = \begin{pmatrix} ca_1 \\ ca_2 \\ \cdots \\ ca_n \end{pmatrix}.$$

   - There are some natural algebraic properties of these operations which will be formally stated later and used to define abstract vector spaces.
   - **Examples.** Consider a linear system given by $2x + 3y = 5, 4x - 3y = 17$.
   - Set $v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, w = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, b = \begin{pmatrix} 5 \\ 17 \end{pmatrix}$.
   - Consider the vector calculation:

$$xv + yw = x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 4x - 3y \end{pmatrix}.$$

   - Thus our linear system can be reinterpreted as a vector equation:

$$xv + yw = b.$$

   - More generally, Consider a system $(A|B)$.
   - Suppose that $A$ has $n$ columns $C_1, C_2, \cdots, C_n$ corresponding to the coefficients of the $n$ variables $x_1, x_2, \cdots, x_n$ respectively,
   - then the equation

$$x_1C_1 + x_2C_2 + \cdots + x_nC_n = B$$

     has the same meaning as the original system of equations.

3. To make it more succinct, we define

   **Def.17: Span of a set of vectors** Given any set $S$ of vectors, we set:

$$Span\ S = \{a_1v_1 + a_2v_2 + \cdots a_mv_m\}$$

   where $m$ is any non negative integer, $v_1, v_2, \cdots, v_m$ are some $m$ vectors in $S$ and $a_1, a_2, \cdots, a_m$ are some scalars.

4. Note that the definition is designed to work for an infinite set $S$, but for a finite set with $n$ elements, we can fix $n = m$.

5. To make our statements even simpler, we now define:

   **Def.18: Matrix times a vector** Given a matrix $A$ with $n$-columns $C_1, C_2, \cdots C_n$, and a vector $v \in \Re^n$ we set

$$Av = a_1C_1 + a_2C_2 + \cdots + a_nC_n \text{ where } v = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{pmatrix}.$$

6. Thus, we can now rewrite the system $(A|B)$ as $AX = B$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$, i.e. $B \in Span\{C_1, C_2, \cdots, C_n\}$.

7. Now we define

   **Def.19: Column Space of a Matrix** For a matrix $A$ with columns $C_1, C_2, \cdots, C_n$ we define $Col(A) = Span\{C_1, C_2, \cdots, C_n\}$.

8. Thus, our consistency condition can be reformulated as $(A|B)$ is consistent iff $AX = B$ has a solution iff $B \in Col(A)$.

9. In view of our earlier consistency condition, this says that $B$ is in $Col(A)$ iff augmenting $B$ to $A$ does not increase its rank!