1 Topics

These are already partly covered in lectures. We collect the details for convenience.

1. Solutions of homogeneous equations $AX = 0$.
2. Using the rank.
3. Parametric solution of $AX = B$.
4. Linear dependence and independence of vectors in $\Re^n$.
5. Using REF and RREF as convenient.

The following topics are not in the book and will be covered over several lectures.

1. Working with Generic Solver $(A|I)$.
2. Reading information from the transformed solver.
3. Using the Generic Solver for consistency conditions.

2 Homogeneous Equations

1. **Def. 20: Homogeneous System of Equations.** A linear system $(A|B)$ is said to be homogeneous when $B = 0$, i.e. the RHS entries in B are all 0.

   In this case, the REF of $M = (A|0)$ can be seen to be $M^* = (A^*|0)$, i.e. the column 0 can be omitted through the reduction process, since it will never change.

2. Clearly, $rank(M) = rank(A)$, so a homogeneous system is always consistent. Indeed, it is also clear that $X = 0$ is a solution to $AX = 0$ and hence consistency is directly obvious!
3. Let the common rank be r. Then there are exactly r pivot variables and $n - r$ free variables.
4. The final solution will consist of solving the pivot variables in terms of the free variables and reporting the conclusion.

Here is an example:

1. Consider a system in REF:

$$\left(\begin{array}{ccccc|c} x & y & z & w & t & RHS \\ \hline 2 & 6 & 0 & -4 & 6 & 0 \\ 0 & 0 & -1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

2. Identify pc list as $(1, 3, 5, \infty)$, pivot variables as x, z, t and the free variables y, w.
3. The fourth equation is ignored. The third gives $t = 0$, the second gives $z =3w + 2t = 3w$ and the first gives $x = -3y + 2w - 3t = -3y + 2w$.
4. The above solution is best reported as a vector:

$$\begin{pmatrix} x \\ y \\ z \\ w \\ t \end{pmatrix} = \begin{pmatrix} -3y + 2w \\ y \\ 3w \\ w \\ 0 \end{pmatrix} = yv_1 + wv_2$$

where

$$v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

Often, it is preferred to replace the original free variables by suitable parameters.

Thus, we may also write:

$$\begin{pmatrix} x \\ y \\ z \\ w \\ t \end{pmatrix} = t_1 v_1 + t_2 v_2$$

We summarize the above results:

1. Thus the solution is seen to be a member of $Span\{v_1, v_2\}$.
2. Denoting a general member of the span as v_h, we write $X = v_h$.

   It is important to remember that v_h stands for any one of an infinite collection of vectors and should not be confused with a specific single vector or with the whole span!
3. In general, if n is the number of variables and $r = rank(A)$ then the solution of $AX = 0$ is always a member of the span of $s = n - r$ vectors $\{V_1, \cdots, V_s\}$.

3 The general case $AX = B$

1. More generally, if we put the augmented matrix $M = (A|B)$ of a non homogeneous system into REF, say $M^* = (A^*|B^*)$. then we need to verify the consistency condition first.

   Recall that the consistency condition is: the rank of the LHS matrix A is the the same as the rank of augmented matrix $(A|B)$.

   If the matrix $(A|B)$ is put in REF $M^* = (A^*|B^*)$, then this amounts to the condition that no row in M^* has pivot in the B^* column. We will develop a "general solver" below, which can easily determine the condition even when we replace the RHS B.
2. If the condition fails, then there is no solution.
3. If the condition holds, then the solution process and reporting is just as above, except the final answer is a fixed vector plus a span of $s = (n - r)$ vectors as before.

Here is an example.

1. Consider a system (already) in REF:

x	y	z	w	t	RHS
2	6	0	−4	6	18
0	0	−1	3	2	−5
0	0	0	0	7	35
0	0	0	0	0	0

2. Identify pc list as $(1, 3, 5, \infty)$, pivot variables as x, z, t and the free variables y, w.
3. The fourth equation is ignored. The third gives $t = 5$, the second gives $z = 3w + 2t + 5 = 3w + 10 + 5 = 3w + 15$ and the first gives $x = -3y + 2w - 3t + 9 = -3y + 2w - 15 + 9 = -3y + 2w - 6$.

4. The above solution is best reported as a vector:

$$\begin{pmatrix} x \\ y \\ z \\ w \\ t \end{pmatrix} = \begin{pmatrix} -3y + 2w - 6 \\ y \\ 3w + 15 \\ w \\ 5 \end{pmatrix} = v_p + t_1 v_1 + t_2 v_2$$

where

$$v_p = \begin{pmatrix} -6 \\ 0 \\ 15 \\ 0 \\ 5 \end{pmatrix}, \; v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

Note that the solution forces $y = t_1$ and $w = t_2$. Sometimes, we may not introduce these new variables, but it is better to bring them in.

5. Note that $Span\{v_1, v_2\}$ is a solution of a related homogeneous equation $AX = 0$.

Thus, $v_h = t_1 v_1 + t_2 v_2$ describes a general member of the solution of the homogeneous equation.

It is important to remember that v_h stands for any one of an infinite collection of vectors and should not be confused with a specific single vector or with the whole span!

6. **Notations** $rownum, colnum$**.** For any matrix A, we shall define $colnum(A)$ to be the number of columns in A and $rownum(A)$ to be the number of rows in A.

7. Suppose that $r = rank(A)$ and $n = colnum(A)$. Set $s = n - r$.

Then we have that the solution of a system $AX = B$ is of the form $X = v_p + v_h$ where v_h is a general linear combination of s solutions of the associated system $AX = 0$.

8. Def. 21: Homogeneous and Particular Solutions. We call v_p as a "particular" solution and v_h as a "homogeneous solution." Note that neither of these are unique, but with proper identifications, they exhibit all the solutions of the system in a parametric form.

9. Def. 22:A homogeneous system $AX = 0$ always has one obvious solution, namely $X = 0$. This is defined to be the "trivial solution." Moreover, as shown above, the system $AX = 0$ has a non trivial solution iff $s = colnum(A) - rank(A) > 0$, or equivalently, there is at least a free variable.

Since the solution of a homogeneous system is of the form $X = t_1 v_1 + \cdots + t_s v_s$ we can say that $v_p = 0$ and $X = v_h$ for a homogeneous system.

4 RREF and its uses

1. We have illustrated how to make and use REF - the "row echelon form". We did not work much with the RREF. Roughly, it needs twice as much work as REF and hence we avoided using it, if it was not really needed. It is, however, needed for some of the later work and we include an illustration of the method to get that form.

Unlike, REF, the form RREF is well defined and thus has theoretical merit. Typically, if we have reached RREF, then the act of "solving equations", becomes, "writing down the answers".

2. A matrix M is said to be in RREF if the following conditions hold:

   - M is in REF.
   - Every row is either a zero row or has pivot entry 1.
   - Pivots of all rows are "lonely" in their columns. This means that the column containing a pivot entry has zero entry in all other rows.

3. An example of RREF

Here is a worked out example of converting REF into RREF.

(a) Consider the following augmented matrix (already in REF) and convert to RREF. Use it to write out the solution of the associated linear system.

(b)

$$\left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 9 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(c) Notice that the pc list is $(1, 2, 3, \infty)$ and the respective pivot entries are $1, 1, 9$.

(d) The process is to start with the last pivot, make all entries above it equal to zero and make it 1.
Then repeat with earlier pivots.

(e) Thus, the operations $R_1 - \frac{3}{9}R_3, R_2 - \frac{6}{9}R_3, \frac{1}{9}R_3$ give:

(f)

$$\left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 3 & 12 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 9 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(g) Then the operation $R_1 - R_2$ finishes off the RREF.

$$\left(\begin{array}{ccc|c} x & y & z & RHS \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(h) The answer to the system can now be simply read off!

5 Linear dependence and independence

1. We have learned the alternate view that if A is a matrix with columns $C_1, \cdots, C_n$ then the equation $AX = B$ is solvable iff B is in the column space $Col(A) = Span(C_1, \cdots, C_n)$.

   We now introduce new concepts which help us decide the nature of solutions more efficiently.

2. **Def. 23: Linearly Dependent vectors** The columns $C_1, C_2, \cdots, C_n$ of A are said to be linearly dependent if the system $AX = 0$ has a non trivial solution, or equivalently $colnum(A) > rank(A)$, i.e. rank of A is less than its number of columns.

3. For future use, we restate this definition more generally thus: **Def. 24(general): Linearly Dependent vectors.** Any set S of vectors is said to be linearly dependent if there is a positive integer n such that n distinct vectors $v_1, v_2, \cdots, v_n$ of S satisfy

   $$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \text{ where at least one of } c_i \text{ is non zero.}$$

   This definition is necessary since in general vector spaces, the vectors may not be columns and we may not be able to make the matrix A from them.

4. Note that this makes sense even for an infinite set S.

5. **Def. 25: Linearly Independent Vectors.** A set S of vectors is said to be linearly independent if it is not linearly dependent!

   A better way to understand this is as follows: If we take any distinct vectors $v_1, v_2, \cdots, v_n$ in S and solve the equation $c_1v_1 + \cdots + c_nv_n = 0$, then it has only the trivial solution $0 = c_1 = \cdots = c_n$.

6. Though logically clear, linear independence can be difficult to verify without better tools. We describe such a tool next.

7. **Convention:** We shall often drop the word "linearly" from the terms "linearly dependent" and "linearly independent".

Tests for dependence/independence

Suppose that we have a set of vectors $v_1, \cdots, v_n$ in some vector space V. If w is a given vector, then the vector equation $x_1 v_1 + \cdots + x_n v_n = w$ is the analog of our linear system of equations.

We describe the corresponding ideas for the abstract vector spaces.

1. First we recall what we know.
   - For a finite set of vectors in $\Re^m$, there is a simple criterion for dependence/independence.
   - Given vectors $v_1, v_2, \cdots, v_n$ in $\Re^m$, make a matrix A by taking these as columns and find its rank (by using REF, for example.).
   - Suppose $rank(A) = r$. It is obvious that $r \leq n = colnum(A)$. Then we have:

     $$v_1, v_2, \cdots, v_n \text{ are linearly dependent iff } r < n$$

     and thus, they are linearly independent iff $r = n$.

6 Generic Solver

1. We discuss a topic which is not in the book, but is a very efficient technique for solving Linear Algebra problems, especially if you have a good calculator handy.

   We learned how to solve a linear system $(A|B)$ for a given right hand side B. Any vector $B \in \Re^m$ can be easily seen to be a well defined combination of special elementary columns

   $$e_1 = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \cdots \\ 0 \\ 0 \end{pmatrix}, e_{m-1} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ 0 \end{pmatrix}, e_m = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}.$$

2. Namely
   $$B = b_1 e_1 + b_2 e_2 + \cdots + b_{m-1} e_{m-1} + b_m e_m.$$

3. It stands to reason that if we solve each of the systems $(A|e_1), (A|e_2), \cdots, (A|e_m)$, then we can write down the complete solution of any $(A|B)$ by simply combining the answers. It would seem like a lot of work, but in reality, it is just as easy as a single system, since the necessary row operations can stay the same.

4. Thus, we set up an augmented matrix $(A|I)$ where I is the "identity matrix" with columns $e_1, e_2, \cdots, e_m$.

5. We then use the row reduction algorithm to change A to its row echelon form (REF) or, even RREF, if desired. Here is what we shall expect to see:

   $$\text{The final form appears as} \left(\begin{array}{c|c} U & U^* \\ \hline 0 & G \end{array} \right)$$

6. The part U has the non zero rows of the REF of our A while 0 below it denotes all its zero rows. Suppose that U has $r = rank(A)$ rows and the last $m - r$ rows are zero.

   The part U^* is simply the transformed part of I across U and G is the important part of the answer in the last $m - r$ rows.

7. We are now ready to handle any given RHS $B = b_1e_1 + b_2e_2 + \cdots + b_{m-1}e_{m-1} + b_me_m$.

   Let $C_1, C_2, \cdots, C_m$ denote the columns of the final RHS $C = \begin{pmatrix} U^* \\ G \end{pmatrix}$.

   It is not hard to see that the REF for $(A|B)$ will have RHS equal to $C \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} = b_1C_1 + b_2C_2 + \cdots + b_mC_m$.

   Further all the entries in the last $m - r$ rows can be shown to be $G \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$ and this must be zero if $(A|B)$ is consistent.

8. **Def. 26: Consistency matrix.**

   The matrix G obtained here is called the "consistency matrix for the system $(A|B)$

   It gives us a simple **Consistency test**, namely: $(A|B)$ is consistent iff $GB = 0$.

   This equation can be interpreted as a condition that B is perpendicular to all the rows of G (transposed into columns).

   Later on, we will see how a complete solution may also be deduced.