**MA322** **Weekly topics and quiz preparations**

Week 4 September 10-14.

# 1 Summary

1. Linear Transformations from $\Re^n$ to $\Re^m$.
2. Matrix of a given transformation.
3. Subspaces associated to a Linear Transformation.
4. Injective (one-to-one) and Surjective (onto) transformations.
5. Transformation with desired properties.

# 2 A Linear Transformation.

1. A Linear Transformation extends the idea of a function so that the domain is $\Re^n$ rather than just the field of real numbers.

   The word "Linear" also means that it has the simplest possible formula consisting of ordinary linear functions.

2. **Def.27: A Linear Transformation** is a map $L : \Re^n \to \Re^m$ satisfying two properties:

   (a) $L(v + w) = L(v) + L(w)$ for all $v, w \in \Re^n$ and

   (b) $L(cv) = cL(v)$ for all $v \in \Re^n$ and $c \in \Re$.

3. The map defined by $L(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x+y \\ x-y \\ 2x+3y \end{pmatrix}$ defines a Linear Transformation from $\Re^2$ to $\Re^3$.

4. **Verification of the definition.** This is checked from definition thus: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then $L(v + w)$ equals:

$$L(\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}) = \begin{pmatrix} v_1 + w_1 + v_2 + w_2 \\ v_1 + w_1 - v_2 - w_2 \\ 2v_1 + 2w_1 + 3v_2 + 3w_2 \end{pmatrix}$$

   which simplifies:

$$\begin{pmatrix} v_1 + v_2 \\ v_1 - v_2 \\ 2v_1 + 3v_2 \end{pmatrix} + \begin{pmatrix} w_1 + w_2 \\ w_1 - w_2 \\ 2w_1 + 3w_2 \end{pmatrix} = L(v) + L(w).$$

5. 

$$L(cv) = L(\begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}) = \begin{pmatrix} cv_1 + cv_2 \\ cv_1 - cv_2 \\ 2cv_1 + 3cv_2 \end{pmatrix} = cL(v).$$

   item**When does the definition fail?** The reason that the calculations work is that the formulas are homogeneous linear expressions in the coordinates of the vectors in $\Re^n$.

6. Thus

$$S(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x+y+1 \\ x-y \\ 2x+3y \end{pmatrix} \text{ and } T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x+y \\ xy \\ 2x+3y \end{pmatrix}$$

   both fail the definition.

   This should be checked.

## 3 Matrix of a Linear Transformation.

1. Here is an example of a map guaranteed to give a Linear Transformation. Let $A$ be a matrix with real entries having $m$ rows and $n$ columns.

   **Def.28: The transformation $T_A$.**

   Define the map $T_A : \Re^n \to \Re^m$ by the formula $T_A(X) = AX$.

2. Then the following calculation shows that $T_A$ is a Linear Transformation.

   $$T_A(v + w) = A(v + w) = Av + Aw = T_A(v) + T_A(w)$$

   and

   $$T_A(cv) = A(cv) = cAv = cT_A(v).$$

   This can be called as the Linear Transformation defined by $A$.

3. **How to find the Matrix of a Linear Transformation?**

   (a) We first need some notation.

   Notation: Define a vector $e_i^n$ to be a column with $n$ entries which are all zero except the $i$-th entry is 1.

   (b) Thus for $n = 3$ we have:

   $$e_1^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

   While working with $\Re^n$ for a fixed $n$, we often drop the superscript $n$ to simplify our display.

4. **The matrix calculated.**

   Given a map $L : \Re^n \to \Re^m$, we calculate the $n$ columns

   $$v_1 = L(e_1^n),\ v_2 = L(e_2^n),\ \cdots,\ v_n = L(e_n^n).$$

   Let $A$ be the matrix with $n$ columns $v_1, v_2, \cdots, v_n$ in order.

5. Then the theorem is: $L$ is a Linear Transformation iff $L(X) = AX$ for all $X \in \Re^n$.

   In other words, $L = T_A$.

6. **Spaces associated with a Linear Transformation..**

   Given any matrix $A$ with $m$ rows and $n$ columns, we have two natural sets associated with it.

   **Def.29: Column and Null spaces of a Matrix.**

   (a) $Col(A) = \{AX \mid X \in \Re^n\}$.

   (b) $Nul(A) = \{X \mid AX = 0 \text{ and } X \in \Re^n\}$.

7. We now consider a Linear Transformation $L : \Re^n \to \Re^m$.

   **Def.30: Kernel and Image of a Linear Transformation.**

   (a) $Image(L) = \{L(X) \mid X \in \Re^n\}$.

   (b) $Ker(L) = \{X \mid L(X) = 0 \text{ and } X \in \Re^n\}$.

8. We shall later define subspaces and show that $Col\ A$ is a subspace of $\Re^m$ and $Nul\ A$ is a subspace of $\Re^n$.

## 4 Relationship with properties of a Linear Transformation.

1. **Def.31: Injective or 0ne-to-one transformation.** A function is said to be one-to-one, if it maps different elements to different elements. For a linear transformation $L$, it is enough to check $L(v) \neq 0$ if $v \neq 0$.

   In other words if $Ker(L)$ contains only the zero vector. In this case, we call the transformation to be injective or one-to-one.

2. **Def.32: Surjective or onto transformation.** A function is said to be onto if every element of the target space is an image of some element.

   For a linear transformation $L : \Re^n \to \Re^m$, the target space is $\Re^m$ and thus the condition reduces to $Image(L) = \Re^m$. In this case, we call the transformation to be surjective or onto.

## 5 The criteria for Injectivity and Surjectivity.

1. When $L = T_A$, we know that $Ker(L) = Nul(A)$ and so we have $L = T_A$ is injective iff $Nul(A) = 0$, or $rank(A) = colnum(A) = n$.

2. When $L = T_A$, we also know that $Image(L) = Col(A)$ and so we have $L = T_A$ is surjective iff $Col(A) = \Re^m$, or $rank(A) = rownum(A) = m$.

3. A Fundamental Fact: The rank of a matrix with $m$ rows and $n$ columns: It is obvious that $rank(A)$ is less than or equal to $\min(m, n)$.

4. This gives some easy but **important conclusions** which are well worth memorizing!

5. Let $A$ be a matrix with $rownum(A) = m$ and $colnum(A) = n$. Let $L = T_A$. Then $rank(A) \leq \min(m, n)$.

6. If $n > m$, then $rank(A) < n = colnum(A)$ and hence $Nul(A)$ is non zero. Consequently, $Ker(L) \neq 0$ and $L$ cannot be injective.

7. If $m > n$, then $rank(A) < m = rownum(A)$ and hence $Col(A)$ is smaller than $\Re^m$. Consequently, $Image(L) \neq \Re^m$ and hence $L$ cannot be surjective.

8. If $n = m$ then $rank(A)$ may be equal to this common value or may be smaller.

   We discuss this next.

9. **Observations continued.**

   If $rank(A)$ equals this common value $n = m$, then the map $L$ is both surjective and injective.

   **Def.33: Isomorphism.** The Linear transformation $L$ is said to be an isomorphism if it is both surjective and injective. In this case, it is a one-to-one onto map from $\Re^n$ to $\Re^n$.

10. If $rank(A) < n = colnum(A)$, then $L = T_A$ is not injective.

11. If $rank(A) < m = rownum(A)$, then $L = T_A$ is not surjective.

12. **Creating a Suitable Linear Transformation.**

    We now show how to use the matrix of a transformation to create a Linear Transformation with a desired property.

13. Suppose we want $L : \Re^3 \to \Re^3$ to rotate all vectors about the $z$-axis.

14. Recall that the vectors $e_1^3, e_1^3, e_1^3$ are along the $x, y, z$ axes respectively.

15. It is clear that we want
$$L(e_1^3) = e_2^3,\ L(e_2^3) = -e_1^3,\ L(e_3^3) = e_3^3.$$

16. Thus $L = T_A$ where $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

17. Now, we can find the rotated image of any desired vector, say $L\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.