Week 5 September 17-21.

1 Summary

1. Matrices and their Operations.
2. Special matrices: Zero, Square, Identity.
3. Elementary Matrices, Permutation Matrices.
4. Voodoo Principle.

2 What is a Matrix.

1. A Matrix is a rectangular array A of numbers (also called scalars, in vector space terminology.) The number of rows is denoted by $rownum(A)$ and the number of columns by $colnum(A)$.

   Convention: We write $A = A_{m\times n}$ to indicate that A has m rows and n columns and may also express this by saying that A has type (or size) $m \times n$.

2. Notation: An entry in the i^{th} row and j^{th} column of a matrix A may be conveniently denoted as $A(i, j)$. Often in books this is written as A_{ij} or even a_{ij} if the author makes a convention of using corresponding small letters for entries.

3. The set of all matrices of type $m \times n$ with entries from a field K will be denoted by $M_K(m, n)$. We may drop the subscript K, if the scalars are already known, e,g, $\Re$.

3 Matrix Operations.

1. We wish to define two operations on $M_K(m, n)$.
2. Scalar Multiplication. Given a matrix A and a scalar $c \in K$, we define a new matrix cA defined by $(cA)(i, j) = c(A(i, j))$.
3. Thus if $c = 5$, then

$$\text{for } A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -5 \\ 2 & 3 & 0 \end{pmatrix} \text{ we have } 5A = \begin{pmatrix} 5 & 0 & 15 \\ 10 & 5 & -25 \\ 10 & 15 & 0 \end{pmatrix}.$$

3.1 Matrix Addition.

1. Matrix Addition. Given $A, B \in M_K(m, n)$, we define $A + B \in M_K(m, n)$ by

$$(A + B)(i, j) = A(i, j) + B(i, j).$$

2. We remark that this is just like the definition in $\Re^n$. Indeed, $\Re^n$ can be thought as a special case of matrices, namely $\Re^n = M_{\Re}(n, 1)$.
3. Thus we get

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -5 \\ 2 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 1 & -3 \\ -2 & 4 & 5 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & 0 & 1 \end{pmatrix}.$$

3.2 Matrix Product.

1. The most complicated and most important operation is the product.
2. Matrix Multiplication. Important: Given matrices A, B, the product AB is defined only if $colnum(A) = rownum(B)$.
3. When this condition is satisfied, suppose that the common number $colnum(A) = rownum(B)$ is equal to s.
4. Then we define
$$(AB)(i,j) = \sum_{k=1}^{s} A(i,k)B(k,j).$$
5. For example, we take matrices $A_{2\times 3}$ and $B_{3\times 2}$ to calculate:
$$AB \;=\; \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 4 & -10 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 15 \\ 13 & 2 \end{pmatrix}.$$

3.3 Comments on the Product.

- We had earlier defined AX when A had type $m \times n$ and X was a column with n entries, i.e. X had type $n \times 1$. This is a special case of our general product.
- It is important to note that if we multiply AB where $A = A_{m\times s}$ and $B = B_s \times n$, then $AB = AB_{m\times n}$.
- Thus, in our example of product we can see that BA would be a matrix of type 3×3. Thus, in general, BA need not be equal to AB.
- Indeed, it is easy to make examples such that AB is defined, but BA is not! For example take, $A_{2\times 2}$ and $B_{2\times 3}$.

4 Some Special Matrices.

- The Zero Matrix 0 is a matrix with all zero entries. To simplify our notation, we often use the same symbol 0 for the scalar zero as well as any sized zero matrix. Its type is deduced from the equation that it fits in.
- A zero matrix serves the purpose of zero in matrix additions.
- Thus $A + 0 = 0 + A = A$ for all A, where 0 is understood to be the same type as A.
- A Square Matrix is a matrix of type $n \times n$ for some positive integer n. These are the only matrices A for which AA is defined!
- The power A^m is defined only for a square matrix A and some positive integer m. It is interpreted as a product of m copies of A.
- Later, we shall extend the definition of A^m to zero or negative integer values of m.

4.1 The Unit Matrix.

- The Unit Matrix I is a square matrix A which has all zero entries except for 1's down the main diagonal. We often specify the type of the unit matrix I by writing I_n, if we have an $n \times n$ matrix.
- For example:
$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
- I behaves like 1 for multiplication. Thus, $AI = IA$ for all matrices A.
- Important: If A is of type 2×3, then the above equation has to be interpreted as $AI_3 = I_2A$.

4.2 The Transpose.

- A natural flipping operation converts all members of $M_K(m,n)$ into $M_K(n,m)$. We define A^T to be the matrix obtained by turning all rows of A into columns. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

- It is easy to verify the following:

$$(A+B)^T = A^T + B^T, \quad (cA)^T = c(A^T).$$

  Also, $(A^T)^T = A$.

- With a little more work we check

$$(AB)^T = B^T A^T.$$

4.3 Symmetric Matrices.

- A matrix A is said to be symmetric if $A^T = A$. Also, A matrix A is said to be antisymmetric if $A^T = -A$.

  Note that A must be square for either of these definitions to hold.

- We leave it as an exercise to prove:

- Let A be any square matrix.

  1. $A + A^T$ is symmetric.
  2. $A - A^T$ is antisymmetric.
  3. Challenge: A can be written as the sum $B + C$ where B is symmetric and C is antisymmetric. Moreover, B and C are uniquely determined by A.

4.4 Elementary Matrices.

- Let $p \neq q$ with $1 \leq p, q \leq n$ and let $c \in K$. we define $E^n_{pq}(c)$ to be the matrix I_n where we have replaced the (p,q)-entry by c.

- Thus, for example

$$E^3_{23}(7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}, \quad E^4_{32}(-4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- Observation: We can remember $E^n_{pq}(c)$ as the matrix obtained from I_n by applying the elementary row operation $R_p + cR_q$.

4.5 Diagonal Matrices.

- A matrix similar to the Identity but slightly different is the diagonal matrix.

- A diagonal matrix $diag(a_1, a_2, \cdots, a_n)$ is a square matrix of type $n \times n$ which has all zero entries, except for the entries $a_1, a_2, \cdots a_n$ on the main diagonal.

- We can see that $diag(a_1, a_2, \cdots, a_n)M$ gives a matrix which is same as M, except its successive n rows are multiplied by the scalars $a_1, a_2, \cdots, a_n$.

- If we multiply the diagonal matrix on the right, then it multiplies the columns instead.

4.6 Permutation Matrices.

- Let $i \neq j$ be chosen with $1 \leq i, j \leq n$.
- We define the matrix P_{ij}^n by swapping the i-th row of I_n with its j-th row.
- Thus,

$$P_{23}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ P_{14}^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

- As before, we drop the superscript n when convenient.
- Multiplication by P_{ij} on the left swaps rows i, j while multiplication on the right permutes columns i, j.

5 Voodoo Principle.

- An important principle of working with matrices is the voodoo principle.
- **Row operations.**

  If you wish to do something to the rows of a matrix M, make a matrix A obtained by performing it on I and then take AM.

  Naturally, choose I so that $colnum(I) = rownum(A)$.
- Example:

  Let

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}.$$

  If you wish to swap its first two rows, then multiply it by P_{12}^3. Thus you can check

$$P_{12}^2 M = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}.$$

5.1 More Voodoo.

- To perform $R_2 - 2R_1$ multiply by $E_{21}(-2)$.

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix}.$$

- To multiply the third row by $\frac{1}{3}$, multiply by $diag(1, 1, \frac{1}{3})$.
- To make the sum of all three rows multiply by $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$.
- **Column operations.** These can be performed by the same idea, except you multiply the prepared matrix on the right.
- Thus $M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ permutes the two columns of M.