MA322 **Weekly topics and quiz preparations**

Week 7-8 October 3-11.

1 Summary

1. Vector space ideas.
2. Change of basis.
3. Matrix of transformation.

2 General Vector Spaces.

1. We briefly discuss axioms for general vector spaces. The best way to understand them is to study examples. You should consult a separate file of notes on vector spaces on this web page.
2. Never forget the basic example $\Re^n$. Also, keep in find new examples of P (the space of all polynomials in one variables over a field - usually the field is $\Re$), function spaces with values in a field, space of matrices of a fixed size and so on.
3. We no longer presume that vectors are columns of numbers, and we no longer simply look at the rank of some matrix to answer questions, except after a careful analysis and argument.
4. We give examples of this next.

3 Reduction to the old case.

1. Consider a real vector space V with a given basis $B = \begin{pmatrix} p & q & r & s \end{pmatrix}$. This means, every vector $v \in V$ has a unique expression
$$v = ap + bq + cr + ds \text{ where } a, b, c, d \in \Re.$$
2. We declare an important notation $[v]_B = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, and say that $[v]_B$ is the coordinate vector of v with respect to the basis B. Remember that B is an ordered basis! A change of order changes the coordinate vector.
3. This coordinate vector satisfies a fundamental identity (to be understood and memorized):
$$\text{For every } v \in V \text{ and a basis } B \text{ of } V, \text{ we have: } \quad v = B[v]_B.$$
4. This defines a linear transformation $L : V \to \Re^4$ defined by $L(v) = [v]_B$.
5. Convince yourself from basic definitions that this L is an isomorphism. So, all concepts of linear dependence, independence etc. about subsets of V can be deduced by taking the image under L.
6. For example consider the following vectors in our vector space V.
$$v_1 = 2p - q + r, v_2 = p + q - 3r, v_3 = p + q + r$$
$$v_4 = q + s, v_5 = p + q + r + s, v_6 = 3p - 2r.$$
7. Using the basis $B = \begin{pmatrix} p & q & r & s \end{pmatrix}$, we calculate the corresponding coordinate vectors:
$$[v_1]_B = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, [v_2]_B = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 0 \end{pmatrix}, [v_3]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$
$$[v_4]_B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, [v_5]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, [v_6]_B = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}.$$

3.1 Various questions.

1. Now we can ask and answer all questions about these vectors.
2. What are the dimensions of $Span\{v_1, v_2, v_3, v_4\}$ and $Span\{v_1, v_2, v_3, v_6\}$?
3. Are the vectors v_1, v_2, v_3, v_4 independent? Do they form a basis of V?
4. Prove that $C = \begin{pmatrix} v_1 & v_2 & v_3 & v_5 \end{pmatrix}$ is a basis of V.
5. Find $[v_4]_C$, $[v_3]_C$, $[v_2 - 5v_3]_C$.
6. Find the dimension and basis for $Span\{v_1, v_2, v_6\}$.
7. Find the dimension and basis for $Span\{v_1, v_2, v_4, v_6\}$.
8. Find the dimension and basis for $Span\{v_1, v_2, v_3, v_4, v_5, v_6\}$.

4 Change of basis.

1. Given a basis $B = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ of an n-dimensional space, recall that we always have:
$$v = B[v]_B \text{ for any } v \in V.$$
2. Given another basis $C = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix}$ we see that $w_i = B[w_i]_B$ for all $i = 1, \cdots, n$.
3. It follows that
$$C = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix} = B \begin{pmatrix} [w_1]_B, [w_2]_B, \cdots, [w_n]_B \end{pmatrix}.$$
Thus we see that
$$v = C[v]_C = B \begin{pmatrix} [w_1]_B, [w_2]_B, \cdots, [w_n]_B \end{pmatrix} [v]_C = BM_B^C[v]_C$$
where $M_B^C = \begin{pmatrix} [w_1]_B, [w_2]_B, \cdots, [w_n]_B \end{pmatrix}$ is called the change of basis matrix from B to C.
4. Thus, $M_B^C[v]_C = [v]_B$.

4.1 Example.

1. Let V have a basis $\begin{pmatrix} u & v & w \end{pmatrix}$ and let $C = \begin{pmatrix} u+v & u+v+w & v+w \end{pmatrix}$. It can be shown that C is also a basis of V.
2. The coordinate vectors with respect to B of members of C are
$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$
3. Verify that
$$M_B^C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$
4. This says that $C = BM_B^C$. Show that $C(M_B^C)^{-1} = B$ or $M_C^B = (M_B^C)^{-1}$.
5. Using it or directly to calculate $[u]_C, [v]_C, [w]_C$.
6. The inverse comes out to be:
$$M_C^B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

7. It follows that for any vector h with $[h]_B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ we get

$$[h]_C = M_C^B \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

8. In particular:

$$[u]_C = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, [v]_C = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, [w]_C = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

5 Matrix of a Transformation.

1. Consider a vectors space V with basis B and vector space W with basis C.
   Explicitly, assume that $B = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ and $C = \begin{pmatrix} w_1 & w_2 & \cdots & w_m \end{pmatrix}$.
2. Let $L : V \to W$ be a linear transformation. We define the matrix of transformation of L with respect to bases B, C to be the matrix $M = \begin{pmatrix} [L(v_1)]_C & [L(v_2)]_C & \cdots & [L(v_1)]_C \end{pmatrix}$.
3. The matrix can also be defined by the property that for all $v \in V$, we have $[L(v)]_C = M[v]_B$.

5.1 Example.

1. Let $V = P_3$ with basis $B = \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix}$ and $W = P_2$ with basis $C = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}$.
   Let $L : P_3 \to P_2$ defined by $L(p(x)) = p'(x) - xp''(x)$.
2. Note that $L(1) = 0, L(x) = 1, L(x^2) = 0, L(x^3) = -3x^2$.
3. Then the matrix of L with respect to the bases B, C is:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

4. Next we show how to use this matrix to calculate properties of L.
5. As above, assume that $L : V \to W$ be a linear transformation and let M be its matrix with suitable bases B, C for V, W respectively.
6. We see that L is injective iff $Ker(L) = 0$ iff $Nul(M) = 0$.
7. We see that L is surjective iff $Im(L) = W$ iff $Col(M) = \Re^m$ where $m = \dim(W) =$ the number of elements in the basis C.
8. Given vectors $u_1, u_2, \cdots u_s \in V$, we can test if their images $L(u_1), L(u_2), \cdots, L(u_r)$ are linearly independent iff the columns $M[u_1]B, M[u_2]_B, \cdots, M[u_r]_B$ are independent.
9. Given a vector $w \in W$ we can find a vector $v \in V$ with $L(v) = w$ thus:
   Solve the equation $MX = [w]_C$. Then $v = BX$ is the answer. Of course, the number of solutions can be $0, 1, \infty$ by the usual theory of equations.

6 Example of using the matrix.

1. Recall the map $L : P_3 \to P_2$ studied earlier. Recall the matrix $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$. Now we decide its properties.
2. Note that $rank(M) = 2$ and thus its columns are dependent and $Col(M)$ is a 2-dimensional space.
   Since $\dim(P_3) = 4 > 2 = rank(M)$ L is not injective. Moreover $Ker(L) = BNul(M)$.

   Since $Nul(M)$ has a basis $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$,

   the $Ker(L)$ has basis $1, x^2$.