MA322 091021 **Solved Quiz 7 (Take Home)** **To be regiven on 10/28/09**

1. Consider the matrices

$$A = \begin{bmatrix} 2 & 3 & 0 & 10 & 11 \\ 1 & 3 & 0 & 6 & 6 \\ 2 & 0 & -1 & 11 & 14 \\ 0 & 6 & 1 & 1 & -2 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 3 & 0 & 2 & 1 \\ 0 & 0 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You are given that M is obtained from A by row transformations. Determine the following quantities, using the matrix M as convenient.

(a) Determine a basis for $Col\ A$. What is the dimension of $Col\ A$?

Answer: Since M is in REF, its rank is clearly 3. Also, its first three columns are independent, since they have the three pivots.

Since M is row equivalent to A, the matrix A also has rank 3 and its first three columns are independent.

So, the $Col\ A$ has dimension 3 with a basis given by the first three columns.

Explicitly, the basis is: $\left(\begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right)$.

(b) Determine a basis for $Nul\ A$. What is the dimension of $Nul\ A$? Carry out row reductions on the matrix A augmented by a zero column. But since a row equivalent M is already given we simply read off the corresponding final equations by using back substitutions.

Thus we have equations in x_1, x_2, x_3, x_4, x_5 to be solved for the pivot variables x_1, x_2, x_3.

- $-x_3 + 3x_4 + 4x_5 = 0$ or $x_3 = x_4 + x_5$.
- $3x_2 + 2x_4 + x + 5 = 0$ or $x_2 = -(2/3)x_4 - (1/3)x_5$.
- $x_1 + 4x_4 + 5x_5 = 0$ or $x_1 = -4x_4 - 5x_5$.

So $Nul\ A$ equals:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_4 \begin{pmatrix} -4 \\ -2/3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -5 \\ -1/3 \\ 4 \\ 0 \\ 1 \end{pmatrix} \mid x_4, x_5 \in \Re \right\} = Span \left\{ \begin{pmatrix} -4 \\ -2/3 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -1/3 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The basis is the pair of vectors giving the span above.

(c) Determine a matrix H such that $Nul\ A = Col\ H$.

This is already solved above. We simply take H to be the matrix with our basis of $Nul\ A$ as columns:

$$H = \begin{pmatrix} -4 & -5 \\ -2/3 & -1/3 \\ 3 & 4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Please turn over

2. Define a linear transformation from P_3 to $\Re^3$ by the formula: $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \\ p(-1) \end{pmatrix}$.

(a) Determine a basis for $Ker(T)$. If we use the standard basis $1, x, x^2, x^3$ for P_3, then the images of these four vectors assembled into columns gives the matrix:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

To find $Ker(T)$ we find $Nul\ M$ and find the preimages of its vectors in P_3.
Row reductions of M augmented by the zero column give:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 8 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 7 & 0 \\ 0 & -2 & 0 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 7 & 0 \\ 0 & 0 & 6 & 12 & 0 \end{pmatrix}.$$

Simple back substitution gives a solution

$$\left\{ x_4 \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix} | x_4 \in \Re \right\} = Span \left\{ \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Lifting back, we get the $Ker(T)$ as the span of $(2 - x - 2x^2 + x^3)$.
Thus, $Ker(T)$ is one dimensional with a basis of one vector

$$(2 - x - 2x^2 + x^3) = (x^2 - 1)(x - 2) = (x - 1)(x - 2)(x + 1).$$

Eureka! Now we can see that we could have bypassed the whole process and simply determined all polynomials which satisfy the conditions to be in the $Ker(T)$, namely

$$p(1) = p(2) = p(-1) = 0.$$

By the remainder theorem, such polynomials have factors

$$(x - 1), (x - 2), (x + 1)$$

and hence must be of the form $c(x-1)(x-2)(x+1)$ or have a basis of one special polynomial!
Note that this only works in P_3. If you are in P_4, then the leftover constant c must be replaced by a linear polynomial and you would get a basis of two polynomials.
You should study what happens in various P_n. Consider $n < 3$ as well as $n > 3$.

(b) Determine a basis for the image of T.
Answer: Since the rank of the matrix M came out 3 equal to its number of rows, clearly its columns span $\Re^3$. Thus, the image of T which equals $Col\ M$ is $\Re^3$.
So, as a basis, you could take the standard basis of $\Re^3$ or use the first three columns of M corresponding to the pivot columns in its REF. Thus, your answer can also be $(T(1), T(x), T(x^2))$.

Shortcut: We will soon learn the fundamental formula

$$\dim(V) = \dim(Ker(T)) + \dim(Image(T))$$

for any linear transformation Ton any vector space V.

Thus, we could have easily deduced the dimension of $Image(T)$ to be $4 - 1 = 3$ and hence concluded that the image is $\Re^3$.

(c) Use your above work to decide if T is one to one (injective) and also if it is onto (surjective).

Answer: We proved that the $Ker(T)$ has dimension 1 and $Image(T)$ coincides with $\Re^3$, so the map is not injective and is surjective.

3. Let V be the subset of $\Re^4$ consisting of all vectors satisfying the conditons:

- The fourth entry equals the sum of the second and the third entry.
- The first entry is the sum of the last three entries.

Prove that V is a subspace of $\Re^4$ by finding a matrix M such that $V = Col\ M$. Show all reasoning.

Answer: If a, b, c, d are the four entries of a vector in V in order, then the given conditions say $d = b + c$ and $a = b + c + d$.

A little thought shows that this means $d = b + c$ and $a = b + c + (b + c) = 2b + 2c$. There are no further conditions on b, c.

So we write:

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2b + 2c \\ b \\ c \\ b + c \end{pmatrix} | b, c \in \Re \right\}.$$

This shows that

$$V = Span \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Hence $V = Col\ M$ where

$$M = \begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Alternative: If you did not "see" the spanning vectors, you could also write

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} | a = b + c + d, d = b + c \right\} = Nul \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix}.$$

This also proves V to be a vector space. You can finish the answer by finding a basis for the Null space by solving the equations.