MA322
080912 **Solved Quizzes 1-3**

Quiz 1

1. For the following system of linear equations in x, y, write down the augmented matrix and carry out the necessary elementary row operations to solve the system.

   Be sure to identify the operations in words or in correct notation introduced in class.

   $$x + 3y = 5, \quad 2x - y = 3.$$

   Answer.
   $$\left[\begin{array}{rr|r} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{rr|r} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right] \quad REF!$$

   Now back substitution gives $y = 1, x = 2$.

2. Give an example of a system of two equations in two variables which does not have a solution (i.e. is inconsistent).

   Both the variables must be present in both your equations.

   You must explain why the equation system has no solution.

   Answer. Consider $x + y = 1, x + y = 2$. First equation subtracted from the second produces an inconsistent equation $0 = 1$. Hence the system is inconsistent.

3. Give an example of a system of two equations in two variables which has infinitely many solutions.

   Both the variables must be present in both your equations and you must have two distinct equations!

   You must explain why the equation system has infinitely many solutions. Be sure to write at least two different solutions to your system.

   Answer. Consider $x + y = 1, 2x + 2y = 2$. The second equation is twice the first, so can be ignored. The first has infinitely many solutions:

   $$x = t, y = 1 - t \text{ with arbitrary } t.$$

   Two concrete solutions are $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$.

Quiz 2

1. Find the general solutions of the system whose augmented matrix is given below. You must reduce the system to REF using the standard algorithm. The final answer may be given by back substitution or producing RREF.

   Be sure to identify the operations in correct notation introduced in class, namely $kR_i, R_i + cR_j$ or P_{ij}.

   $$\left[\begin{array}{rrrr|r} 1 & -7 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] \xrightarrow{R_3 + R_1} \left[\begin{array}{rrrr|r} 1 & -7 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 2 & 12 \end{array}\right] \xrightarrow{R_3 + 4R_2} \left[\begin{array}{rrrr|r} 1 & -7 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & -6 & 0 \end{array}\right] \quad REF!$$

   This can be solved by back substitution.

   $$x_4 = 0, x_3 = -3, x_1 = 7x_2 + 5 \text{ with } x_2 \text{ arbitrary.}$$

2. Write down a vector equation equivalent to the given system of equations in the above question.

$$x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -7 \\ 0 \\ 7 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix}.$$

3. You are given:

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 8 \\ -2 \end{pmatrix}, v_3 = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} \text{ and } W = Span\{v_1, v_2, v_3\}.$$

Answer the following questions.

- Does the vector $\begin{pmatrix} 10 \\ 3 \\ 3 \end{pmatrix}$ belong to W? Justify your answer.

  We try to decide if the family of vectors $m_t = \begin{bmatrix} 10 \\ 3 \\ 3+t \end{bmatrix}$ is in W for various values of t. This amounts to solving the system:

  $$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3+t \end{array}\right] \xrightarrow{R_2+R_1/2, R_3-R_1/2} \left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2+t \end{array}\right] \xrightarrow{R_3+R_2/4} \left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0+t \end{array}\right]$$

  Thus any non zero value of t makes the vector m_t outside W, since the equations are inconsistent.
  For $t = 0$ the equations are consistent, so the given vector m_0 is in W.

- Is $W = \Re^3$? Either prove that $W = \Re^3$ or find at least one explicit vector in $\Re^3$ which is not in W. **Answer.** Already answered. The vector m_1 is not in W.

Quiz 3

1. Determine the condition on a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be in

$$Span \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Answer. Convert the following matrix to REF.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 2 & 2 & 4 & c \end{array}\right] \xrightarrow{P_{12}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b \\ 0 & 1 & 1 & a \\ 2 & 2 & 4 & c \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b \\ 0 & 1 & 1 & a \\ 0 & -2 & -2 & c-2b \end{array}\right] \xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & c-2b+2a \end{array}\right]$$

So the consistency condition is $c - 2b + 2a = 0$. Once this is satisfied, our matrix is in REF and has solutions.

2. Give an example of four vectors in $\Re^3$ which span $\Re^3$. You must prove why your vectors must span $\Re^3$. **Answer.** Take

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and let v be any vector in $\Re^3$.

The $Span\{e_1, e_2, e_3\} = \Re^3$ since any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ can be written as $ae_1 + be_2 + ce_3$.

So $Span\{e_1, e_2, e_3, v\}$ is also $\Re^3$ since the last vector is already in $Span\{e_1, e_2, e_3\} = \Re^3$.

3. Consider a homogeneous system of linear equations $AX = 0$ where A is an $m \times n$ matrix. Suppose that the REF of A has r pivots.

   State and explain a condition based on r, n and m which guarantees that the system has a non trivial solution. Be sure to explain the condition by citing appropriate results from the book.

   Answer. The system is consistent since $X = 0$ is a solution. We have a system of m equations in n variables.

   The number of free variables is $n - r$ and the condition for a non trivial solution is that there is a free variable or $r < n$.

   Use your condition to prove the following:

   - If $m < n$, then the above linear system always has a non trivial solution.

     Answer. We know that $r \leq \min(m, n)$. From the given condition we get $r \leq m < n$ so our condition for non trivial solutions is satisfied.

   - If $m \geq n$, then the above system could have only a trivial solution. Give an explicit example (with justification) of a system with $m = 3 = n$ which has only a trivial solution.

     Answer. Consider the matrix

     $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

     Since it is already in RREF, its rank r is clearly 3. So, $m = n = r = 3$ and there is no free variable, so $AX = 0$ has only a trivial solution.

4. Suppose that A is an $m \times n$ matrix where $m > n$. Explain why the columns of A cannot span $\Re^m$. Be sure to cite appropriate results from the book.

   Answer. Here, we have $r \leq \min(m, n) = n < m$. The condition for the columns of A to span $\Re^m$ is that the rank r equals m (or a pivot in every row in REF).

   Since $r < m$ the columns of A cannot span $\Re^m$.