MA322
081020 **Solved Quizzes 4-6**

Quiz 4

Consider the following matrices and answer the following questions.

$$A = \begin{bmatrix} 5 & -5 & 2 \\ -1 & 4 & 5 \\ -4 & -4 & -2 \end{bmatrix},\ B = \begin{bmatrix} 2 & 5 \\ -3 & 3 \\ 4 & -4 \end{bmatrix},\ C = \begin{bmatrix} 5 & 2 & 4 \\ 3 & 5 & -5 \end{bmatrix},\ D = \begin{bmatrix} -3 & -1 \\ -3 & 2 \end{bmatrix}.$$

Either calculated the indicated quantities or explain why it is not defined. For example, to explain why AD is not defined, you would write: "$colnum(A) = 3 \neq 2 = rownum(D)$."

$D^2 + D$ answer: $\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$3A - 5I$ answer: $\begin{bmatrix} 10 & -15 & 6 \\ -3 & 7 & 15 \\ -12 & -12 & -11 \end{bmatrix}$

AB answer: $\begin{bmatrix} 33 & 2 \\ 6 & -13 \\ -4 & -24 \end{bmatrix}$

BC answer: $\begin{bmatrix} 25 & 29 & -17 \\ -6 & 9 & -27 \\ 8 & -12 & 36 \end{bmatrix}$

AC answer: Undefined, since $colnum(A) = 3 \neq 2 = rownum(C)$

$2B + 5C$ answer: Undefined since B, C have different sizes.

$BD - DB$ answer: Undefined since DB is undefined because $colnum(D) = 2 \neq 3 = rownum(B)$

Quiz 5

1. Find the inverse of the matrix M given below. Use the inverse to calculate the solution to the equation $MX = B$ for the matrix B given below. **Do not apply Gaussian elimination again to $(M|B)$. Using the inverse is the main point! Be sure to write out the final answer. I will not search for it all over the page.**

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

Answer:

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}.$$

Also, the solution is

$$X = M^{-1}B = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

2. State the basic formula relating to a square matrix A, its adjoint A^{adj}, its determinant $\det(A)$ and its inverse A^{-1}. Use this formula to find the adjoint M^{adj} using your answer above. **Do not start using the definition of the adjoint.**

   Answer: The formula is:

$$A^{-1} = \frac{A^{adj}}{\det A} \text{ provided } \det A \neq 0.$$

   Here $\det M^{-1} = -1$ as is evident due to its triangular nature. (For a triangular matrix, the answer is the product of diagonal entries. If you need to swap rows to get to a true triangular form, then you multiply the answer by -1 for each row swap.

3. The following determinants can be calculated by a simple manipulation or observation. Calculate their values while explaining your reasoning. If you try to use expansions, look for using rows or columns with lots of zeros.

   (a)

$$\begin{vmatrix} 20 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 6 & 12 & -2 & 0 \end{vmatrix}$$

   Swapping first two and last two rows produces a lower triangular matrix with diagoanl entries $2, 3, -2, 1$. Their product is -12. Since we need an even number of swaps, it is also the final answer.

   (b)

$$\begin{vmatrix} 3 & 3 & 8 & 1 \\ 2 & 6 & 0 & 0 \\ 6 & 6 & 16 & 2 \\ 2 & 10 & 1 & 0 \end{vmatrix}$$

   Begin a row reduction to make zero entries and $R_3 - 2R_2$ produces a zero third row. So the determinant is zero.

(c)

$$\begin{vmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

Swapping second and third rows produces a diagonal matrix with diagonal entries $5, 2, 7, 3$ whose product is 210. Since we used an odd number of swaps, the answer is -210.

(d)

$$\begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 10 \\ 0 & 2 & 1 & 15 \\ 3 & 1 & 0 & 20 \end{vmatrix}$$

Swapping first and fourth rows and also the second and the third rows produces an upper triangular matrix with diagonal entries $3, 2, 7, 5$ whose product is 210. Since we used an even number of swaps, the final answer is also 210.

Quiz 6

1. Let V be a subset of $\Re^4$ consisting of all vectors satisfying the conditions:

   - The first entry equals the sum of the second and the third entry.
   - The fourth entry is the sum of the first three entries.

   Prove that V is a subspace of $\Re^4$ by finding a matrix M such that $V = ColM$. Show all reasoning.

   If we take a typical vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ then the given conditions say that $x_1 = x_2 + x_3$ and $x_4 = x_1 + x_2 + x_3$.

   Solving these two equations (by matrix method or simple manipulation) we see that x_2, x_3 can be taken as free variables and we have $x_1 = x_2 + x_3$ and $x_4 = 2x_2 + 2x_3$.

   Thus we have:

$$M = \left\{ \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \\ 2x_2 + 2x_3 \end{bmatrix} | x_2, x_3 \in \Re \right\}.$$

   Collecting coefficients, we deduce:

$$M = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} | x_2, x_3 \in \Re \right\} = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Therefore,

$$M = Col \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}.$$

2. Consider the matrices

$$A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You are given that M is obtained from A by row transformations. Determine the following quantities, using the matrix M as convenient.

(a) Determine a basis for $Col\ A$. What is the dimension of $Col\ A$? Since M is in REF we see that its rank is 3 and its pivot columns are $1, 2, 4$. So the corresponding columns in A give a basis of $Col\ A$.

Thus:

$$\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$$

is the answer. The dimension of the space is thus 3.

(b) Determine a basis for $Nul\ A$. What is the dimension of $Nul\ A$? Solve $AX = 0$ or equivalently $MX = 0$. The pivot variables are x_1, x_2, x_4 and x_3, x_5 are free.

The solutions are

$$x_1 = 2x_3 - 7x_5, x_2 = -\frac{5}{2}x_3 + \frac{1}{2}x_5, x_4 = -4x_5.$$

Thus a general solution is given as:

$$X = x_3 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ \frac{1}{2} \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Thus dimension is 2 and a basis for $NulA$ is

$$\begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ \frac{1}{2} \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

(c) Determine a matrix H such that $Nul\ A = Col\ H$. Simply take the matrix whose columns are the two vectors above.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}.$$

Determine a matrix H such that $Col\ A = Nul\ H$.

Answer: We show two methods.

Generic method. Solve the augmented matrix below using row reductions.

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 2 & 2 & 4 & b \\ 3 & 3 & 6 & c \end{bmatrix}.$$

Just two operations $R_2 - 2R_1, R_3 - 3R_1$ produce:

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 0 & 0 & b-2a \\ 0 & 0 & 0 & c-3a \end{bmatrix}.$$

Thus vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfy the consistency conditions $b - 2a = 0, c - 3a = 0$. This can be written as:

$$\begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So the column space of the given matrix is the null space of $H = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

Transposition method. If our column space is the null space of some matrix H, then each row X of H multiplied to the given A must produce a zero row, i.e. $XA = 0$. Taking transpose, we get $A^T X^T = 0$. Thus, X^T is in $Nul\ A^T$.

We calculate $Nul\ A^T$ by the usual method. Thus, we solve for X^T using the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right]$$

Clearly, the second and third rows become zero after $R_2 - R_1$ and $R_3 - 2R_1$. Thus we are left with a single equation $x_1 + 2x_2 + 3x_3 = 0$. Treating x_2, x_3 as free variables we see that the solutions are written as:

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus, these solutions form the column space of the matrix

$$\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Remembering that the answers were transposes of the X above, we see that our matrix H is the transpose:

$$\begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Thus the summary of this method is as follows. If you wish to write $Nul\ A$ as $Col\ H$ for some H, find out $Nul\ A^T$ by solving the augmented system $(A^T|0)$.

Write your solution as column space of some matrix as usual. The transpose of the matrix is the desired H.