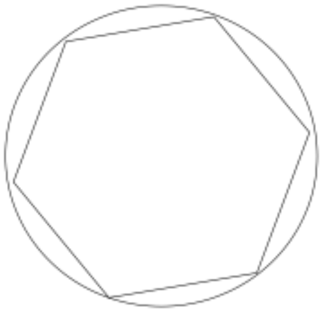
Approximating Square Roots

MA 330

Archimedes



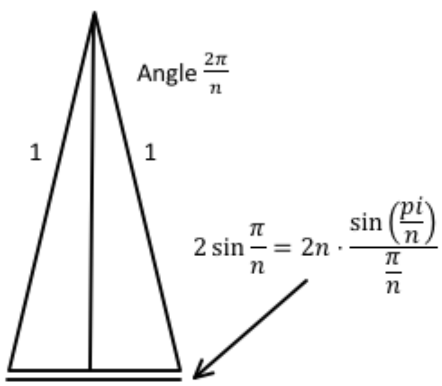
Attempting to calculate the perimeter of an n-sided polygon, P_n.

Modern Interpretation

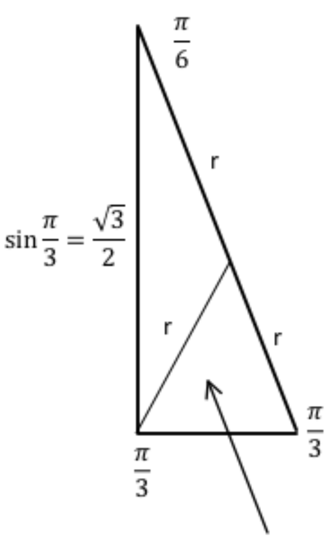
Using a limit would be how this would be done in the modern day.

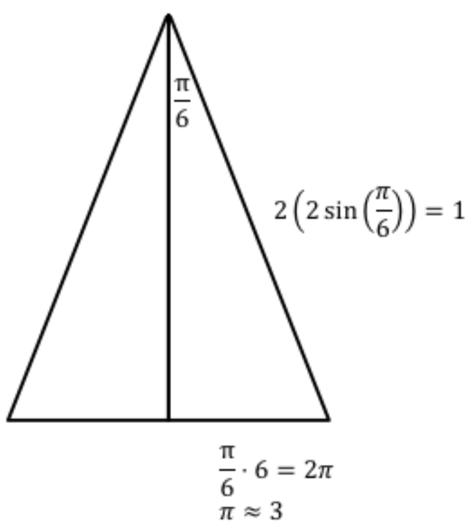
$\lim_{n\to\infty} P_n = 2\pi r$

Using Geometry



Known Quantities





This forms an equal-lateral triangle.

Using the methods above, the approximate value of π was determined to be 3, since for a 6-sided polygon exampled above would mean $n = 6 \rightarrow 2\pi = 6 \rightarrow \pi = 3$.

What about $n = 12$?

$$n = 12 \rightarrow \sin\frac{\pi}{12} \rightarrow \sin\left(\frac{2\pi}{6}\right) = 2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$$

To calculate the above, the double-angle identity must be used for cosine.

$$\cos\frac{\pi}{6} = 1 - 2\sin^2\frac{\pi}{12}$$

$$= \sqrt{1 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{\sqrt{3}}{2}$$

This is why there was the need for an efficient way to approximate square roots.

Approximating Square Roots

The idea is to approximate a square root in terms of a rational number $\frac{p}{q}$.

$$2^2 - 3(1^2) = 1 \rightarrow \left(\frac{2}{1}\right)^2 - 3 = \left(\frac{1}{1}\right)^2$$

Suppose $(2 - 3\sqrt{3})(2 + 3\sqrt{3}) = -2^2 - 3 = 1$

$$(2 - \sqrt{3})^2 = 2^2 - 2(2\sqrt{3}) + 3 = 7 - 4\sqrt{3}$$
$$7 - 4\sqrt{3} \rightarrow (7 - 4\sqrt{3})(7 + 4\sqrt{3}) \rightarrow 7^2 - 4^2 3 = 1$$

So what does this do?

$\frac{7}{4} \approx \sqrt{3}$ with an error of $\frac{1}{4^2}$.

General Form

$$(p_1 + q_1\sqrt{D}) = p_1p_2 + p_1p_2 + (p_1q_2 + p_2q_1)\sqrt{D}$$

This form was used in the 7th century by Brahmaqupta of India, he called it "bhārana".
$(p_1, q_1)(p_2, q_2) \rightarrow (p_1p_2 + q_1q_2, p_1q_2 + p_2q_1)$

This is similar to complex numbers.
$a + bi, c + di \rightarrow ac - bd + (ad + bc)2$
Where $ac - bd = i^2$

Each time the formulae gives an answer of 2 numbers (p and q), they can be plugged back in to obtain a more accurate pair of numbers.

This was called **Pell's Equation** even though Pell didn't do anything with it, originally Euler referred to it in this way and it caught on. In the 17th century Fermat challenged his colleagues to find a value for x and y to solve for $D = 61$. His colleagues Lagrange, Brucker, and Wallis all gave long solutions based on continued fractions.